1.18 Trial and Improvement

- Whether a problem is sensible to solve by trial and improvement depends on what other methods the pupils have met. Using trial and improvement to find numbers like $\sqrt{20}$, for example (as some textbooks do), is pointless for pupils who know about the square root button. Similarly, solving quadratics by trial and improvement will be unmotivating for pupils who know about the formula.
  In general it's better to choose problems that are difficult to solve by other methods. The only difficulty then is that the teacher has no easy access to the answers! It's possible to use graphical calculators or spreadsheets to find the solutions, and on the following page are some accurate solutions to various cubic and quadratic equations.

1.18.1 One way to begin is by plotting a graph of $y = x^2 + x$ for $0 \le x \le 5$ and $0 \le y \le 30$.

x	0	1	2	3	4	5
y	0	2	6	12	20	30

Answer: 2.70156212…

This task is suitable for pupils who don't know a method for solving quadratic equations. If the pupils have already covered this, use a cubic equation instead.

If we know x , working out y is easy.
But going the other way is hard.
If $y = 10$, what is x ?
It's hard to work out (try it by algebra), but easy to check if we think we know the answer.

Lots of things are like this: public key encryption works on the principle that it's easy to encode something but very hard to decode it unless you know the key.
Differentiation is usually much easier than integration.

Pupils can read off an approximate value from the graph.
Could draw a more accurate graph from, say, $x = 2.5$ to $x = 3$ to get a better approximation.

But we could work it out without bothering to draw the graph. We'd need to assume that there's a continuous curve throughout the region.

x	2		3
y	6		12

too small   too big

Using a horizontal table such as this is more visually helpful for some pupils than a normal table of rows. It has similarity to a number-line.

Use this to find x to 2 dp when $y = 22$.

Answer: 4.21699057… or 4.22 (2 dp) – not 4.21

Can you find a negative solution, also to 2 dp?

Answer: -5.21699057… or -5.22 (2 dp)

1.18.2 Some thought needs to be given to how to get an answer to a particular degree of accuracy; say, 3 dp.

For example if the answer is between, say, 4.216 and 4.217, we just need to try 4.2165; if 4.2165 is too small then the answer is 4.217 to 3 dp.
Otherwise the answer is 4.216 to 3 dp.
This works because if 4.2165 is too small then whatever the answer is it is between 4.2165 and 4.217, and all those numbers round to 4.217 to 3 dp.

1.18.3 Can you invent a graph where this method *wouldn't* find an answer?
(Here the graph doesn't go as low as $y = -2$.)

A discontinuous curve will cause problems.
Otherwise the only problem will be cases where there is no solution; e.g., $x^2 + x = -2$.

Solutions to Cubic Equations

The method of Cardano (1545) for solving $x^3 + mx = n$ gives

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

This leads to the values in the table below. These can be used to check trial and improvement work.
Always request a particular degree of accuracy (e.g., 3 dp) for answers.

	equation	***x***
1	$x^3 + x = 1$	0.682327804
2	$x^3 + 2x = 1$	0.453397652
3	$x^3 + 3x = 1$	0.322185355
4	$x^3 + 4x = 1$	0.246266172
5	$x^3 + x = 2$	1.000000000
6	$x^3 + x = 3$	1.213411663
7	$x^3 + x = 4$	1.378796700
8	$x^3 + 2x = 2$	0.770916997
9	$x^3 + 2x = 3$	1.000000000
10	$x^3 + 2x = 4$	1.179509025

	equation	***x***
11	$x^3 - x = -1$	–1.324717957
12	$x^3 - x = 1$	1.324717957
13	$x^3 - x = 2$	1.521379707
14	$x^3 - x = 3$	1.671699882
15	$x^3 + 10x + 5 = 0$	–0.488353313
16	$x^3 = 5x + 10$	2.905474006
17	$x^3 + 12x = 10$	0.791942868
18	$x^3 + 5x = 7$	1.119437527
19	$x^3 = 11 + x$	2.373649822
20	$x^3 + 6x - 9 = 0$	1.206959814

Solutions to Quadratic Equations

The solutions below come from using the equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

so there are two solutions (x_A and x_B) for each equation.

The problems can be set as equations in x to solve or they can be converted into words; e.g., the equation $x(x+3) = 30$ is equivalent to the question "the product of two numbers is 30 and the difference between the numbers is 3, what are the two numbers?"; e.g., areas of rectangles given the difference between the lengths of the sides. Answers should be given to a particular degree of accuracy (e.g., 3 dp).
Depending on the context, negative answers may or may not be acceptable.

	equations		x_A	x_B
1	$x(x+3) = 30$	$x^2 + 3x - 30 = 0$	$x = 4.178908346$ $x+3 = 7.178908346$	$x = -7.178908346$ $x+3 = -4.178908346$
2	$x(x+10) = 50$	$x^2 + 10x - 50 = 0$	$x = 3.660254038$ $x+10 = 13.66025404$	$x = -13.66025404$ $x+10 = -3.660254038$
3	$x(20-x) = 50$	$-x^2 + 20x - 50 = 0$	$x = 2.928932188$ $20-x = 17.07106781$	$x = 2.928932188$ $20-x = 17.07106781$
4	$x(x+2) = 25$	$x^2 + 2x - 25 = 0$	$x = 4.099019514$ $x+2 = 6.099019514$	$x = -6.099019514$ $x+2 = -4.099019514$
5	$x(x+1) = 10$	$x^2 + x - 10 = 0$	$x = 2.701562119$ $x+1 = 3.701562119$	$x = -3.701562119$ $x+1 = -2.701562119$
6	$x(2x+1) = 12$	$2x^2 + x - 12 = 0$	$x = 2.21221445$ $2x+1 = 5.424428901$	$x = -2.71221445$ $2x+1 = -4.424428901$