1.19 Sequences

- With some pupils it may be worth avoiding statements like "now we're going to do algebra" because of the reputation algebra has for being "hard"! You can say later, "you've been doing algebra!"
- It's necessary to distinguish between *term-to-term* rules plus starting value(s) (i.e., *inductive* definitions like $u_n = n+3;\ u_1 = 1$) and *position-to-term* rules (i.e., *deductive* definitions like $u_n = 3n-2$, the same sequence).
  Neither is necessarily "better"; they're just different ways of describing sequences.
- Pupils sometimes assume that if a rule works for the first few terms then it will always work. (This is probably because we often use such straightforward sequences.) "Points and Regions" (section 1.19.11) is a helpful investigation to do to show that this isn't always the case.
- *Proving* a rule needs some insight into what is going on. A clear diagram often helps, or looking at the problem from another angle. Comments of this kind appear on the right below.
  Pupils often confuse *proving* a result for *all values* of n with *checking* a result for a *few particular values* of n.

1.19.1 "James, give me a number between 1 and 10."
Whatever the pupil says, the teacher doubles and adds 1 and writes it on the board.
Eventually ask, "What's going on?"
Can use quite difficult functions.
e.g., $y = 3x-1$, $y = 10-x$, $y = 42+x$

Teacher as "function machine".

Another system is for the teacher to have a red board pen and the volunteer pupil a black pen. The pupil records the pupils' numbers in black in the left column of the table; the teacher records the teacher's number in red in the right column of the table.
After a while we decide that it would be better to choose black numbers in order, so next time instead of choosing the black numbers randomly we just write 1-5 in order (and the pupil can sit down!).

black	*red*
4	*17*
1	*2*
7	*32*

Using black and red colours to represent sets of numbers is helpful.
You can eventually write, e.g.,
black = 5 × red – 3 and then b = 5 × r – 3 and then $b = 5r - 3$.

Pupils can invent their own table of numbers or you can present one on the board:

black	red	yellow	green	purple	white
1	11	5	16	3	8
2	12	10	22	8	11
3	13	15	28	13	14
4	14	20	34	18	17
5	15	25	40	23	20

It's worth keeping the colours going for a bit, but soon you can start saying things like "I don't have a white pen, so I'll write 'white' in black" and pupils will soon accept the "colour" even if it's all written in black.
(This work is slightly easier on a blackboard than a whiteboard because you generally have more different colours available.)

Avoid embarrassing a pupil who is colour-blind.

Look for connections – always horizontal, not down the columns. "I want a statement about how one colour is connected to another colour", "How do you get from the black numbers to the red numbers?", etc.

e.g.,
(or write using function machines or in words)
$r = b+10$ *(not* $b+1$*);* $y = 5\times b$; $g = r+y$;
$p = y-2$; $w = \frac{g}{2}$; *etc.*

Pupils can make up their own tables of numbers with rules for getting from one colour to another.
(See "Making Formulas" sheet.)

Can you get from b to w?
× 3 then + 5; could write using function machines or perhaps $w = 3b+5$.

These formulas can then be "formalised" into $w = 3y+2$, *etc.*

1.19.2 "Matchstick problems", or similar.
Readily available in books. Pupils have to find a

Need to establish that n *stands for the "pattern number", so* $n = 1$ *for the first drawing, and so on,* formula to give the number of lines (or matchsticks) used in a sequence of drawings.

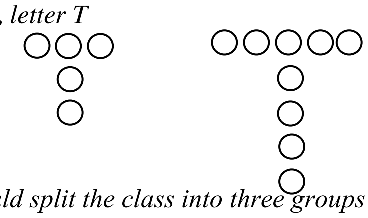
and that u_n (or whatever symbol) represents the n^{th} term; i.e., how many matchsticks are used in the n^{th} pattern.

1.19.3 **NEED** linking cubes. Cube Animals.
Make a simple dog/horse/giraffe, etc. from about 8 linking cubes. Make a second "larger" one of the same "species" but using more cubes. What would the third one look like? How many cubes would it need for its legs?, etc.
What would the 100th one look like? If you had 100 cubes and you wanted to make the biggest model you could, how many would you use for the head?

You can use different colours for different parts of the animal (e.g., red for the head, blue for the body, green for the legs, yellow for the tail) and work out formulas for each colour; e.g., if n is the model number then the number of cubes in the head (red) could be $2n$, perhaps.
If you add the expressions for the number of cubes in the different body parts you get the formula for the total number of cubes needed for the whole animal.

1.19.4 Investigate letters of the alphabet made out of squares or circles; e.g., AEFHILNTVY.
Different pupils may use different rules about how to get the next letter (some may be enlarging, and others stretching), but that needn't matter.

e.g., letter T



1.19.5 A general method for finding the n^{th} term involves looking for common differences.
The argument here is that if the numbers go up in 3's, say, then the sequence must have something to do with the 3 times table (because the numbers in the 3 times table go up in 3's).
There are 3 different sequences of integers that go up in 3's: the 3 × table; the 3 × table shifted on 1; and the 3 × table shifted on 2. (The 3 times table shifted on 3 is the same as the 3 × table except that the first term is lost.)
It's sensible to call these $3n$, $3n+1$ and $3n+2$ (or $3n-1$).
(See "Finding the Formula" sheet.)

Could split the class into three groups and "chant" each of these sequences separately; then put them together to see that it makes 1, 2, 3, 4, 5, 6, 7, 8, ...

"Finding the Formula" answers:

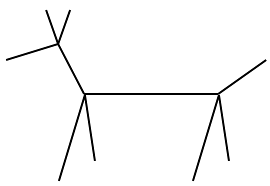
1a	$t=7n$	***1b***	$t=5n-1$
2a	$t=3n-2$	***2b***	$t=n+14$
3a	$t=10n+5$	***3b***	$t=2n+22$
4a	$t=5n+8$	***4b***	$t=n-1$
5a	$t=4n+308$	***5b***	$t=3n-2.5$
6a	$t=4n+997$	***6b***	$t=1000n+1$
7a	$t=6n+1$	***7b***	$t=2$
8a	$t=5-n$ *or* $t=-n+5$	***8b***	$t=28-2n$

1.19.6 There are lots of interesting investigations to do at this stage; e.g., "Loops" (see sheet, although sheet certainly not necessary).

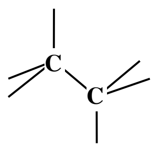
Answers to "Loops":
c increases by 1 each time,
and $a=2c$ and $b=c+1$.

A similar one is "Stick-Animals".
Like "stick-people", these are made from short straight lines joined together at their ends. No loops are allowed.
e.g., stick-dogs:

Drawings must be clear.



Count the number of lines (10), the number of junctions (3) and the number of "ends" (8).
What is the connection between these numbers?



Some similarity with chemical molecular structures (aliphatic hydrocarbons – compounds containing only carbon and hydrogen atoms and no rings – acyclic);
e.g., ethane is C_2H_6; the two carbon atoms are like junctions and the six hydrogens like ends, and the number of chemical bonds is 2+6–1=7.

junctions + ends = lines + 1

1.19.7 Quadratic, Cubic and Beyond.
A linear sequence: $u_n = an+b$

$$a+b \qquad 2a+b \qquad 3a+b$$
$$a \qquad\qquad a$$

A quadratic sequence: $u_n = an^2+bn+c$

If the first *differences are constant, then the sequence is* linear, *and $u_n = an+b$ (see left). You can work out a and b because a is the common difference and the first term is $a+b$, or you can find the "zeroth" term and that is b.*

If the first differences aren't *constant, you find the*

$a+b+c \qquad 4a+2b+c \qquad 9a+3b+c$

$3a+b \qquad 5a+b$

$2a$

A similar method will work for a cubic sequence $u_n = an^3 + bn^2 + cn + d$ (or higher polynomial sequences).

An alternative is to write, say for a cubic, $u_n = a + b(n-1) + c(n-1)(n-2) + d(n-1)(n-2)(n-3)$.

second differences. If they're constant, then the sequence is quadratic, and $u_n = an^2 + bn + c$ (see left). You can work out a, b and c because the common (second) difference is $2a$ (so you can work out a). You can use this value of a together with the first of the first differences ($3a+b$) to find b and then use this with the first term ($a+b+c$) to find c.

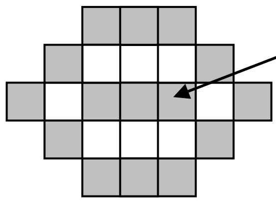
You substitute $n=1$ to find a, $n=2$ to find b and so on, but then you have to expand the brackets and simplify to get your final formula.

1.19.8 Ever-Increasing Rectangles.
Start with 3 squares in a line (shaded at the centre of the diagram on the right) and surround them with other squares (just 8 more, touching each side but not at the corners – the white ones on the right).
Keep going, surrounding the shape you have at each stage with squares on the edges.
Record the number of squares in each layer.

(It's a very good idea to shade in alternate layers so that they don't get muddled up.)

Investigate different shaped rectangles.
How many squares would be in the n^{th} layer surrounding an x by y rectangle?

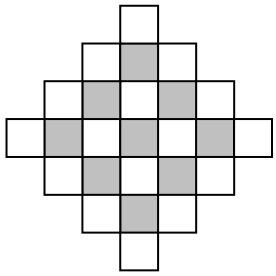
Can make nice display work.



For a 1 × 3 rectangle, the n^{th} layer has $4n+4$ squares.
In general, for an x by y rectangle, the n^{th} layer will have $4(n-1)+2(x+y)$ squares.
A drawing makes it clear that the first term comes from the "corners", and the second from the "sides".

The total number of squares after n layers is given by $2n^2 - 6n + 4 + 2(n-1)(x+y) + xy$.

If you start with a 1 × 1 square ($x = y = 1$), then a simpler pattern emerges (see right). If you look at the diagram at 45°, you can see that the total number of squares is the sum of two square numbers (3 × 3 = 9 grey squares and 4 × 4 = 16 white squares).
In general, the total number of squares after n layers is $n^2 + (n-1)^2 = 2n^2 - 2n + 1$.
(The first square counts as layer 1.)



Can extend by trying hexagons instead (on isometric paper). Or you can use other non-rectangular shapes made of squares.
(It works best if the hexagons you begin with join side-to-side, not corner-to-corner, so if you want to use the paper in landscape orientation, then you need to have the dots the other way round from normal – hence the "unusual" dotty paper included – see sheet.)

For a row of x hexagons, the n^{th} layer contains $6n + 2(x-1)$ hexagons, and the total number of squares after n layers is $(3n-2)(n-1) + (2n-1)x$.

Or you can extend to cubes *in 3-dimensions. For n 3-dimensional "shells", the total number of cubes is $\frac{1}{3}(4n^3 - 6n^2 + 8n - 3)$.*

1.19.9 Dots in Rectangles.
Draw a 3 × 4 rectangle on some 0.5 cm × 0.5 cm squared paper. (You could use square dotty paper, but it isn't necessary)
How many dots are there inside it? (Count a dot as any place where the grid lines cross).

There are 6 dots, and it's easy to generalise that an x by y rectangle will contain $(x-1)(y-1)$ dots.

What about tilted rectangles?
Try ones that are at 45° to start with.

The other sides have a gradient of –1. In general if one pair of sides have gradient m, the other pair will

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Call them "gradient 1" because the sides with positive gradient have a gradient of 1.

Try rectangles of width 2 (this means 2 diagonal spaces) and find a formula for the number of dots inside as the length of the rectangle changes.
Find an overall formula for an x by y "gradient 1" rectangle.

Extend to rectangles of gradient 2, 3 and m.

have gradient $-\frac{1}{m}$, but that's another investigation!

For "gradient 1" and $y=2$, $n=3x-1$.

For "gradient 1" rectangles, $n=2xy-x-y+1$.
For "gradient 2", $n=5xy-x-y+1$.
For general "gradient m" rectangles,
$n=(1+m^2)xy-x-y+1$.
Notice how all these formulas are symmetrical in x and y (if you swap around x and y you get the same formula).

1.19.10 Try the differences method on the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, …
(Each term is the sum of the two previous terms.)

Which other sequences will behave like this when you try to find common differences?

Answer: doubling sequences like
1, 2, 4, 8, 16, 32, 64, … or 5, 10, 20, 40, 80, …

Every set of differences are just the Fibonacci sequence again. Although the term-to-term rule is simple ($u_n=u_{n-1}+u_{n-2}$), the n^{th} term is given by

$$u_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right],$$

and this is fairly hard to prove.
Fibonacci (1170-1250) wrote a famous book called Liber Abaci.

1.19.11 Points and Regions.
If 10 points are spaced evenly around the circumference of a circle, and every point is joined to every other point with a straight line, how many pieces is the circle divided into?
Start with a small number of points and do BIG drawings (A4 paper at least). Put a cross in those areas you've counted so you don't get muddled up. (If you need to re-count the same drawing, use a different colour for the crosses.) Work out a formula for the number of regions you get if you begin with n points.

The reason why is that the total number of regions must be 1 + number of chords + number of intersections inside the circle.
There are ${}^nC_2$ chords because each chord has two ends and ${}^nC_4$ intersections because each is defined by 4 points on the circle. This gives the formula $1+{}^nC_2+{}^nC_4$, which is the same.

Pupils could invent a sequence that looks as though it follows as simple pattern but deviates later; e.g., $\frac{1}{6}(n^3-6n^2+29n-6)$ goes 3, 6, 9, **13, 19, 28,** …

This is a good task to use to emphasise the need not to make assumptions too early on!

no. of points	*no. of regions*
1	*1*
2	*2*
3	*4*
4	*8*
5	*16*
6	***31***
7	***57***

This is about as far as you can get on an A4 sheet of paper counting carefully, but it's enough data.

Note 31 and not 32. The pattern **isn't** 2^{n-1}.

4th differences turn out to be constant (1), so the equation is quartic.

The solution is that for n points the number of regions = $\frac{1}{24}(n^4-6n^3+23n^2-18n+24)$.

Answer for 10 points is 256. (Ironically this is a power of 2; actually 2^8.)

For further details, see the sheet, which illustrates a different way of obtaining a formula from a number sequence. Although more work, this method shows how each term can be worked out separately.

1.19.12 Triangles.
(See related investigation in section 1.17.3.)

If n straight lines intersect, what is the maximum possible number of triangles created?

Answer: any 3 lines could make a triangle, so it will be the number of ways of choosing 3 lines out of n; in other words,

$${}^nC_3=\frac{n(n-1)(n-2)}{3!}=\frac{1}{6}(n^3-3n^2+2n).$$

1.19.13 Frogs. (A well-known investigation.)
Have 4 boys and 3 girls sitting on chairs in a row with an empty chair in between.

B B B B _ G G G

They are the "frogs". They have to swap around so

Getting the minimum number of moves isn't always easy. Recording the pattern of slides and jumps helps to see what to do next.
Pupils could state their conclusions about how to get the fewest moves; e.g.,

- *The pattern should always have some symmetry; e.g., for 4 boys and 3 girls it is* that the final arrangement is

```
G G G _ B B B B
```

The only two types of move allowed are

- a *slide* of 1 place (only) into an adjacent empty seat; and
- a *jump* over 1 person (only) into an empty seat.

The question is can it be done, and if so what is the minimum possible number of moves.
What if the numbers of boys and girls change?

Pupils can use counters or cubes (blue for boys, green for girls) and record the moves on squared paper.

You can record results in a 2-way table:

number of girls g \ number of boys b	1	2	3	4
1	3	5	7	9
2	5	8	11	14
3	7	11	15	19
4	9	14	19	24

(There are other sequences of moves when $b = 1$ that are also a minimum.)

Notice that the table is symmetrical about the diagonal, and the formulas are symmetrical so that if you swap b for g you get the same formula.

Note that this doesn't prove that such a minimum sequence of moves (i.e., with nobody wasting a move by going backwards) will be possible; only that if it is this is how many moves it will take.

1.19.14 Rectangles in Grids (see sheet).

1.19.15 Stopping Distances (see sheet).
(The figures have been adjusted slightly to fit a quadratic formula.)

Answers:

1. *We imagine that the driver has a fixed (average) "thinking time", regardless of the car's speed, so if travelling twice as fast, he'll cover twice the distance in that thinking time.*
2. *20 mph = 32 kph = 9 m/s, so thinking time = $6 \div 9$ = about 0.7 seconds. This would probably represent quite an alert driver.*
3. *When the speed doubles, the braking distance goes up by a factor of about 4. This happens*

SJSJJSJJJSJJJSJJSJS (19 moves).
(S = slide, J = jump)

- *No boy or girl ever needs to go "backwards".*
- *Try starting by sliding what you have most of and then jumping and sliding (once each) the other.*
- *Always jump after a slide, and jump as much as possible (because a jump moves each person twice as far as a slide but still only counts as one move)!*

Let b = number of boys, g = number of girls and n = minimum number of moves.
In general,

$$n = (b+1)(g+1) - 1$$
$$= bg + b + g$$

So when $b = g$, $n = (b+1)^2 - 1$; i.e., 1 less than the next square number.

If $b = 1$, then it's fairly easy to see that $n = 2g + 1$. First the boy slides, then the 1st girl jumps, then the boy slides again and the 2nd girl jumps and so on until the g th girl jumps ($2g$ moves by then – each girl has jumped once and the boy has slid g times). Finally the boy has to slide once to get to the end seat. So altogether $2g+1$ moves (g slides and $g+1$ jumps).

When $b > 1$ and $g > 1$, it's harder to justify the number of moves necessary. It's possible to argue like this: There must be a total of bg jumps (each boy has to jump over or be jumped over by each girl). From the start, each girl has to move $b+1$ places to get to her final position, and each boy has to move $g+1$ places (the +1 because of the empty seat they all have to pass). So the minimum total number of shifts must be $g(b+1) + b(g+1) = 2bg + b + g$. But the bg jumps will make up $2bg$ shifts (each jump moves a frog 2 places), so the number of slides must be the remaining $b + g$. So the total number of moves must be $bg + b + g$ of which bg are the jumps and $b + g$ are the slides.
Very rich investigation.

Answers (continued):

4. *If v is the speed (mph), then in metres,*
   thinking distance = $\frac{3v}{10} = 0.3v$;
   braking distance = $2\left(\frac{v}{10}\right)^2 - \frac{2v}{5} + 8$
   $= 0.02v^2 - 0.4v + 8$; and
   overall stopping distance = $2\left(\frac{v}{10}\right)^2 - \frac{v}{10} + 8$
   $= 0.02v^2 - 0.1v + 8$.
5. *At 80 mph, overall distance = 128 metres.*
6. *The thinking distance would be about the same, but the braking distance could be ten times as*

because kinetic energy ($\frac{1}{2}mv^2$) is proportional to the square of the speed v (m is the mass), and assuming that the brakes apply a constant slowing force they will need 4 times as much distance to reduce the kinetic energy to zero.

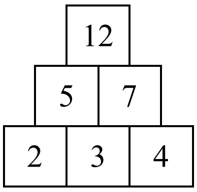
much, or even more.

7. The condition of the car (tyres, brakes, mass, etc.) and the driver (tired, alcohol, distractions, experience, skill, etc.).

1.19.16 Pyramids. (A well-known investigation.)

Pyramid shapes are made out of a triangle number of rectangular boxes.
Each rectangle contains a number equal to the sum of the numbers in the two rectangles underneath it.
The numbers in the bottom row are consecutive integers.
Work out a way to predict the number in the rectangle on the top (the peak number) given the numbers at the bottom for different heights of pyramids.

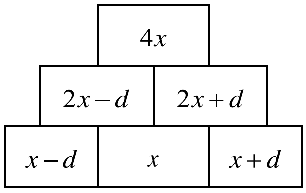
e.g., How can you predict 12 given (2,3,4) without working out the other numbers?



Hint: Start with *odd* numbers of rows.

(See sheet of blank pyramids that may save time.)

Answer: 12 = 4 × 3.
In general, for a 3-row pyramid, the peak number = $4x$, where x is the middle number on the bottom row.



This works even if the bottom row aren't consecutive numbers, so long as they go up with a common difference (d above).

To deal with larger pyramids (with odd numbers of rows), treat them as containing this three-row unit. The middle number of a row goes up 4 times when you jump up 2 rows.
For n (odd) rows, peak number = $2^{n-1}m$.

Even-height pyramids (n even) need considering separately. There is no middle number at the bottom this time.
In general, if x is the smallest number *in the bottom row (so x is the number at the* end *now, not in the middle), and d is the common difference along the bottom row, then the peak number = $\frac{1}{4}2^n(2x+d(n-1))$. In fact this works whether n is odd or even so long as x is the smallest number in the bottom row.*

1.19.17 **NEED** Graphical Calculators.
Use the `ANS` feature to generate different sequences; e.g., 5, 8, 11, 14, …

Linear sequences are easy, but you can get the square numbers, for example, if you start with 1 and use the inductive formula $u_n = (\sqrt{u_{n-1}} + 1)^2$.

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Making Formulas

Here are some numbers. Each column of numbers is called by the name of a colour.

black	red	yellow	green	purple	white
1	3	2	4	6	8
2	6	3	7	9	11
3	9	4	10	12	14
4	12	5	13	15	17
5	15	6	16	18	20

There is a connection between the black numbers and the red numbers.
If you multiply a black number by 3 you get the corresponding red number.
We can write this as a formula or using number machines.
Look at line 1 in the list below.
Complete the list.

	Formulas	**Number Machines**
1	red = 3 × black	black → [**× 3**] → red
2	yellow =	black → [**+ 1**] → yellow
3	green =	red → [ ] → green
4	green =	black → [**×**] → [**+**] → green
5	purple =	yellow → [ ] → purple
6	purple =	red → [ ] → purple
7	purple =	green → [ ] → purple
8	purple =	black → [ ] → [ ] → purple
9	white =	purple → [ ] → white
10	white =	green → [ ] → white
11	white =	yellow → [ ] → [ ] → white
12	white =	red → [ ] → white

Extra Tasks

- Write some more formulas connecting these colours.
- Try to write a formula that starts **black =**

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Making Formulas ANSWERS

Here are some numbers. Each column of numbers is called by the name of a colour.

black	red	yellow	green	purple	white
1	3	2	4	6	8
2	6	3	7	9	11
3	9	4	10	12	14
4	12	5	13	15	17
5	15	6	16	18	20

There is a connection between the black numbers and the red numbers.
If you multiply a black number by 3 you get the corresponding red number.
We can write this as a formula or using number machines.
Look at line 1 in the list below.
Complete the list.

	Formulas	**Number Machines**
1	red = **3 × black**	black → **× 3** → red
2	yellow = ***b* + 1**	black → **+ 1** → yellow
3	green = ***r* + 1**	red → **+ 1** → green
4	green = **3*b* + 1**	black → **× 3** → **+ 1** → green
5	purple = **3*y***	yellow → **× 3** → purple
6	purple = ***r* + 3**	red → **+ 3** → purple
7	purple = ***g* + 2**	green → **+ 2** → purple
8	purple = **3*b* + 3**	black → **× 3** → **+ 3** → purple
9	white = ***p* + 2**	purple → **+ 2** → white
10	white = ***g* + 4**	green → **+ 4** → white
11	white = **3*y* + 2**	yellow → **× 3** → **+ 2** → white
12	white = ***r* + 5**	red → **+ 5** → white

Extra Tasks

- Write some more formulas connecting these colours.
- Write a formula that starts **black =** for example, **black = yellow – 1**

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Finding the Formula

Work out formulas that fit the numbers in these tables.
Check that your formulas work by trying them when $n = 3$.

1

a

n	1	2	3	4
t	7	14	21	28

b

n	1	2	3	4
t	4	9	14	19

2

a

n	1	2	3	4
t	1	4	7	10

b

n	1	2	3	4
t	15	16	17	18

3

a

n	1	2	3	4
t	15	25	35	45

b

n	1	2	3	4
t	24	26	28	30

4

a

n	1	2	3	4
t	13	18	23	28

b

n	1	2	3	4
t	0	1	2	3

5

a

n	1	2	3	4
t	312	316	320	324

b

n	1	2	3	4
t	0.5	3.5	6.5	9.5

6

a

n	1	2	3	4
t	1001	1005	1009	1013

b

n	1	2	3	4
t	1001	2001	3001	4001

7

a

n	1	2	3	4
t	7	13	19	25

b

n	1	2	3	4
t	2	2	2	2

8

a

n	1	2	3	4
t	4	3	2	1

b

n	1	2	3	4
t	26	24	22	20

Extra Task Make up a table of numbers like this that fit a more tricky formula.

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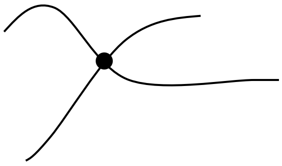
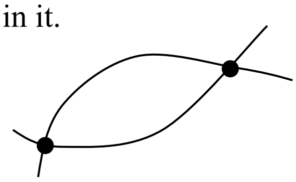
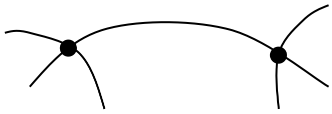
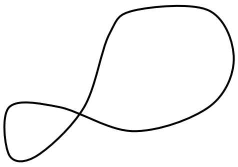
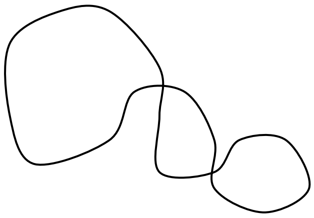
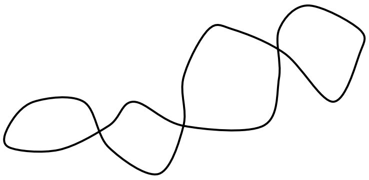
Loops

This task is about **crossovers**, **blobs** and **arcs**.

A **crossover** is a point where 2 lines cross.	A **blob** is an area with lines around the outside and nothing in it.	An **arc** is a line joining one crossover to another.
		

Look at the shapes below.
Complete the pattern by drawing the loops in boxes 4 to 9.
For each shape, count the number of crossovers, blobs and arcs.

Write your results in the boxes underneath. Number 1 has been done already.

1			**2**			**3**		
								
a	***b***	***c***	***a***	***b***	***c***	***a***	***b***	***c***
2	2	1	4					
4			**5**			**6**		
a	***b***	***c***	***a***	***b***	***c***	***a***	***b***	***c***
7			**8**			**9**		
a	***b***	***c***	***a***	***b***	***c***	***a***	***b***	***c***

What do you notice from your results?
Can you explain the pattern in the numbers?

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Points and Regions (Finding a Possible Formula)

n = number of points $\qquad$ r = number of regions

It turns out that the 4th differences are all 1, so the highest term in n must be $\frac{1}{24}n^4$. Using a spreadsheet we can calculate this amount and subtract it from r. Then we work out the new 1st, 2nd and 3rd differences. This time the 3rd ones are constant, and that gives us the term in n^3.

n	r	$\frac{1}{24}n^4$	$r-\frac{1}{24}n^4$	1st diff	2nd diff	3rd diff	$-\frac{1}{4}n^3$
1	1	0.042	0.958				–0.250
2	2	0.667	1.333	0.375			–2.000
3	4	3.375	0.625	–0.708	–1.083		–6.750
4	8	10.667	–2.667	–3.292	–2.583	–1.500	–16.000
5	16	26.042	–10.042	–7.375	–4.083	–1.500	–31.250
6	31	54.000	–23.000	–12.958	–5.583	–1.500	–54.000
7	57	100.042	–43.042	–20.042	–7.083	–1.500	–85.750
8	99	170.667	–71.667	–28.625	–8.583	–1.500	–128.000

Now we have the first two terms, we calculate $r-\frac{1}{24}n^4+\frac{1}{4}n^3$, which ought to be no more than quadratic. 2nd differences are constant at $1\frac{11}{12}$, so we have the term in n^2. Finally, subtracting this we obtain a linear sequence which we can write as $-\frac{3}{4}n+1$, and we've finished.

n	r	$r-\frac{1}{24}n^4+\frac{1}{4}n^3$	1st diff	2nd diff	$\frac{23}{24}n^2$	$r-\frac{1}{24}n^4+\frac{1}{4}n^3-\frac{23}{24}n^2$	1st diff	$-\frac{3}{4}n$	$r-\frac{1}{24}n^4+\frac{1}{4}n^3-\frac{23}{24}n^2+\frac{3}{4}n$
1	1	1.208			0.958	0.250		–0.750	1
2	2	3.333	2.125		3.833	–0.500	–0.750	–1.500	1
3	4	7.375	4.042	1.917	8.625	–1.250	–0.750	–2.250	1
4	8	13.333	5.958	1.917	15.333	–2.000	–0.750	–3.000	1
5	16	21.208	7.875	1.917	23.958	–2.750	–0.750	–3.750	1
6	31	31.000	9.792	1.917	34.500	–3.500	–0.750	–4.500	1
7	57	42.708	11.708	1.917	46.958	–4.250	–0.750	–5.250	1
8	99	56.333	13.625	1.917	61.333	–5.000	–0.750	–6.000	1

So we have $r=\frac{1}{24}n^4-\frac{1}{4}n^3+\frac{23}{24}n^2-\frac{3}{4}n+1$ or $r=\frac{1}{24}(n^4-6n^3+23n^2-18n+24)$. Using this formula we can predict values of r for larger n.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
r	1	2	4	8	16	31	57	99	163	256	386	562	794	1093	1471	1941	2517	3214	4048	5036

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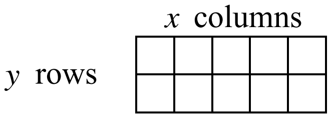
Rectangles in Grids

How many ways are there of fitting an $a \times b$ rectangle into an $x \times y$ grid, where a, b, x and y are all positive integers, and $\max(a,b) \le \min(x,y)$?
Every vertex of the rectangle must lie on a grid point, and the sides of the rectangle must be parallel to the grid lines.



- Start with rectangles that are 2×1.
  How many ways can you fit these rectangles into an $x \times y$ grid?

  Think about where the middles of the rectangles will go.
  There will be y rows of $x-1$ "horizontal" rectangles.
  There will be x columns of $y-1$ "vertical" rectangles.
  So the total number of rectangles, n, will be $n = y(x-1) + x(y-1) = 2xy - x - y$.
  e.g., for $x \times 2$ grids, $n = 3x - 2$;
  for $x \times 3$ grids, $n = 5x - 3$; etc.

- Now try rectangles that are $a \times 1$.
  Similar reasoning gives $n = y(x-a+1) + x(y-a+1) = 2xy - ax - ay + x + y$.
  This works provided $a \ne 1$.

- Now try rectangles that are $a \times b$.
  This time,
  $$\begin{aligned} n &= (y-b+1)(x-a+1) + (x-b+1)(y-a+1) \\ &= 2xy - ax - bx - ay - by + 2ab - 2a - 2b + x + y + 2 \end{aligned}, \text{ assuming } a \ne b.$$

It's possible to extend this investigation to 3 dimensions, where you're fitting little cuboids into big cuboids (lattices).

The number of ways of fitting an $a \times b \times c$ cuboid into an $x \times y \times z$ cuboid is

$$\begin{aligned} n = &(z-c+1)(y-b+1)(x-a+1) \\ &+(z-c+1)(y-a+1)(x-b+1) \\ &+(z-b+1)(y-a+1)(x-c+1) \\ &+(z-b+1)(y-c+1)(x-a+1) \\ &+(z-a+1)(y-b+1)(x-c+1) \\ &+(z-a+1)(y-c+1)(x-b+1) \end{aligned}$$

This assumes that $\max(a,b,c) \le \min(x,y,z)$ and that none of a, b and c are equal to each other (otherwise symmetry will mean that there are fewer ways).

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Stopping Distances

These are average stopping distances for an ordinary car on a good road surface.

speed (mph)	thinking distance (m)	braking distance (m)	overall stopping distance (m)
20	6	8	14
30	9	14	23
40	12	24	36
50	15	38	53
60	18	56	74
70	21	78	99

Answer these questions.

1. *Thinking distance* is the time it takes for the driver to realise he/she needs to press the brake pedal and to get his/her foot onto the pedal.
   It's proportional to speed, so it doubles as the speed doubles.
   Why do you think that is?

2. What "reaction time" do these numbers assume that the driver has?
   Would you react that quickly?

3. *Braking distance isn't* proportional to speed.
   What happens to braking distance as the speed goes up?
   Why do you think that is?

4. Find formulas for
   - the *thinking distance* in terms of the speed;
   - the *braking distance* in terms of the speed;
   - the *overall stopping distance* in terms of the speed.

5. Some people think that the speed limit on motorways and dual carriageways should be raised to 80 mph.
   Use your formula to predict the overall stopping distance at 80 mph.

6. How much more do you think these figures would be in wet or icy conditions?

7. In real life, what factors apart from speed and weather conditions will affect the overall stopping distance?

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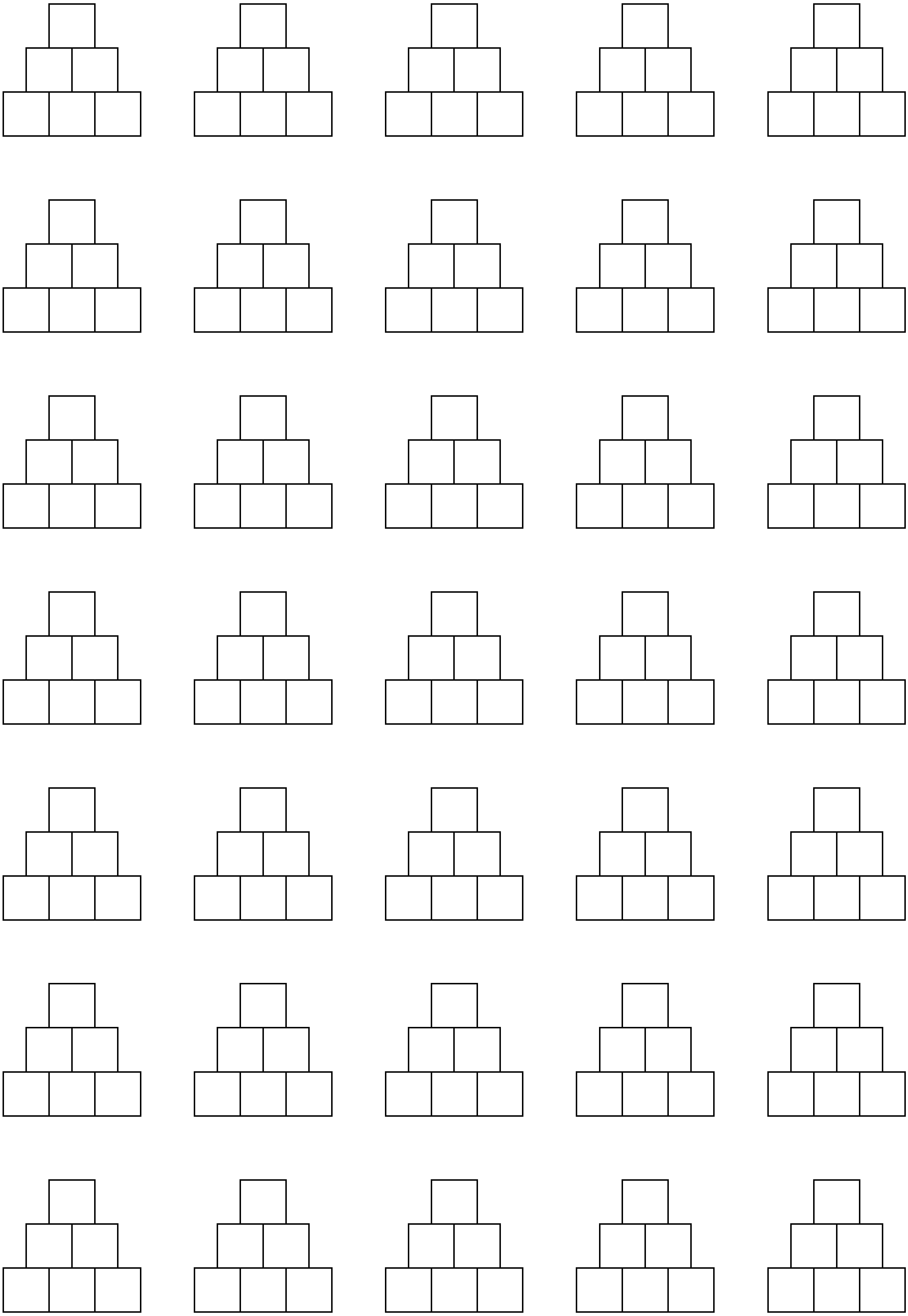

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