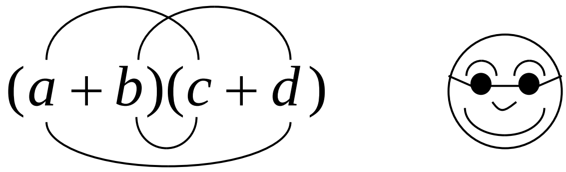
1.20 Formulas, Equations, Expressions and Identities

- **Collecting terms** is equivalent to noting that 4 + 4 + 4 + 4 + 4 + 4 can be written as 6 × 4; i.e., that multiplication is repeated addition. It's wise to keep going back to numbers. Algebra is just generalised numbers: if it works with numbers, it works with algebra.
  Key questions are "Do numbers do that?", "Is that how numbers behave?"
  This is probably more helpful than treating letters as "objects"; e.g., with $3x+5x$ rather than asking "how many x's have we got altogether?" ask "how many *lots of* x have we got altogether?"
- Part of the problem of **simplifying** is knowing what counts as "simpler". Need to know what's impossible; e.g., that you can't simplify a^2+3b (you can't combine "unlike" terms). Where indices are involved, it sometimes helps temporarily to put the × signs back in; e.g., to simplify $3a^2b\times 4ab^3$, rewrite it as $3\times a\times a\times b\times 4\times a\times b\times b\times b$ (or imagine this), then remember that when multiplying you can change the order without it affecting the answer, so this equals $3\times 4\times a\times a\times a\times b\times b\times b\times b=12a^3b^4$.
- **Expanding** – getting rid of the brackets. For expanding pairs of binomials, some pupils use the **FOIL** acronym as a reminder that you get four terms: **f**ront, **o**utside, **i**nside and **l**ast. One way of illustrating this is to draw curves to link both terms in the *first* bracket with both terms in the *other* bracket.



  And this gives a "smiley face", which is easy to refer to: e.g., "Have you done the nose?"
- **Factorising** – Just as with numbers, this means finding something that goes into it, although in algebra it can be "whole expressions" rather than whole numbers. So this is the opposite of expanding: "Now we've done all that work getting rid of the brackets, we want to work out how we can put them back in again!"
- **Balancing** – "ALWAYS DO THE SAME THING TO BOTH SIDES" makes a good title for a lesson!
  This is probably clearer than "moving" things from one side to the other, because it seems confusing that a positive term on the left becomes negative on the right when the two sides are supposed to be equal to each other.
- **Rearranging formulas** – This is a bit like solving equations; trying to find x (to make it the subject), say, in terms of the other quantities rather than as a numerical answer. It's helpful, again, to begin numerically with something like $24=2\times 10+4$ and to ask "Can we write an expression for 10 using the other numbers?": $10=\frac{24-4}{2}$.

1.20.1	Words that mean different things in maths from what they mean in ordinary life or in other subjects. Think of some examples. *Experimental constants are of a different kind.*	*Take-away, volume, factor, negative, net, etc.* *Also expression and identity.* *Discuss the difference between an unknown, a variable and a constant. (There are different sorts of "constants"; e.g.,* π *is absolute;* $g=9.81$ *m/s²* *varies from place to place.)*
1.20.2	Which do you think ought to be larger 456 × 458 or 457 × 457? What about other numbers?	*Answer: the second one (by 1), because* $(n+1)(n-1)\equiv n^2-1$. *Works even for negative numbers as well.*
1.20.3	**NEED** "Alphabet Code" sheets. A code like this one is probably better than using $a=1$, $b=2$, $c=3$, … (although that is easier to remember) because it avoids pupils getting the idea that those letters always stand for those numbers.	*Answers: 1. "Well Done"; 2. "I Love Algebra"; 3. "Maths is Fun"; 4. "Fish and Chips".* *Pupils can make up their own coded messages for each other using the same letter code.* *("Remember I know the code, so don't say anything you wouldn't want me to read!")*
1.20.4	Balancing. **NEED** acetate (see pages) of scales and little weights and objects to slide around to create different problems. "I have a weight problem!"	*It may be wise initially to praise any sequence of steps for solving a linear equation even if it isn't the shortest or most elegant. So long as you're doing the same thing to both sides each time, strategy can*

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Pupils come to the projector to solve the problem by "always doing the same thing to both sides". They take off weights from both sides – eventually you can see what the mystery object must weigh.

Key questions are: "What are you going to do to both sides?", "What do you need to do to the left side?", etc.

kg is included in the "weights" so that the value of an object (its weight) isn't confused with the number of objects; i.e., the value of x *isn't confused with the co-efficient of* x.

The numbers are really mass *(amount of matter) rather than* weight *(heaviness), but this is an unnecessary complication here.*

develop later.

It's probably worth setting out equation-solving very carefully: only one equals sign per line; all equals signs lined up vertically; curly arrows down the right side (or both sides) saying what happens between each step and the next;
e.g.,

$$\begin{aligned} 12-2x &= x+3 && \quad +2x \\ 12 &= 3x+3 && \quad -3 \\ 9 &= 3x && \quad \div 3 \\ 3 &= x \end{aligned}$$

(Don't include kg in the equation-solving itself.) It's better to have curly arrows between *lines so that they don't get muddled up with the algebra* on *the lines.*

1.20.5 The numbers 2 and 2 have the strange property that 2 + 2 = 2 × 2. Find another pair of equal numbers that do the same thing.

Answer: $ab = a+b$ *means that* $a(b-1) = b$ *and* $a = \frac{b}{b-1}$. *Clearly* $a = 2$, $b = 2$ *satisfy this, as do* $a = 0$, $b = 0$.

What if the numbers don't have to be the same as each other?
What numbers are possible?
(The numbers don't have to be integers.)
What if 3 is one number – what does the other number have to be to make this work?

If they don't have to be equal, then there are infinitely many pairs of numbers that will work. The only number that neither can ever be is 1. Either both numbers are > 1 or else one number is < 0 and the other is between 0 and 1. (There is also the possibility that both numbers are 0, as mentioned above.)
e.g., 3 and $1\frac{1}{2}$ *or – 2 and* $\frac{2}{3}$.

What about with three or more numbers? When does sum = product?

With three numbers, the only integers that work are 1, 2, 3.

1.20.6 Painted cube.
A large cube is made out of 125 small cubes. The large cube is painted on the outside. When it's dismantled and the small cubes are examined separately, how many have paint on 3 sides, 2 sides, 1 side and no sides?
Do any cubes have more than 3 sides painted?
What about an $n \times n \times n$ cube?

Answers: no cube has paint on more than 3 sides.

sides painted	***5 × 5 × 5 cube***	$n \times n \times n$ ***cube***
0	*27*	$(n-2)^3$ *[the ones inside]*
1	*54*	$6(n-2)^2$ *[6 faces]*
2	*36*	$12(n-2)$ *[12 edges]*
3	*8*	8 *[8 corner cubes]*
total	*125*	n^3

It's a good algebra exercise to check that the total of the right column is n^3 *(see working on the left).*

Total of right column

$$\begin{aligned} &= 8+12(n-2)+6(n-2)^2+(n-2)^3 \\ &= 8+(n-2)(12+(n-2)(6+(n-2))) \\ &= 8+(n-2)(12+(n-2)(n+4)) \\ &= 8+(n-2)(12+(n^2+2n-8)) \\ &= 8+(n-2)(n^2+2n+4) \\ &= 8+n^3+2n^2+4n-2n^2-4n-8 \\ &= n^3 \end{aligned}$$

1.20.7 What famous formulas do you know or have you heard of, perhaps in school?

They will most likely come from science.

Answers: (some possibilities)

- $F = ma$ *: Newton's (1642-1727) 2nd Law, where* F *is the resultant force in a particular direction acting on a particle of mass* m *and producing an acceleration of magnitude* a *in the same direction;*

Because c^2 is so large, if even a tiny mass of matter is converted into energy, a huge amount of energy is released.

The chance of finding the particle at a particular spot is proportional to ψ^2 at that spot (that's the meaning of ψ).

- $E = mc^2$ *: Einstein's (1879-1955) famous equation from Relativity Theory, stating the equivalence of energy E and mass m, where c is the speed of light;*
- $$\frac{-h^2}{8\pi^2 m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+V\psi = E\psi$$
  (Schrödinger's (1887-1961) equation from Quantum Mechanics for the wave function ψ for a particle of mass m and energy E moving in a potential of V, where h is Planck's (1858-1947) constant (6.626 × 10^{-34} J s).

1.20.8 Which formulas do you know that are interesting or useful to you?

Examples may come from hobbies, cooking, walking, thunder and lightning, etc.

A teacher's one is
concentration span (mins) = age (years) + 2

Do they agree with this one?
("Sorry, what were you saying?"!)

1.20.9 **NEED** Cards (see two sheets – back-to-back photocopy onto coloured card that is thick enough for the numbers not to show through).
Hold up a letter card (e.g., f) facing the class. Look at the number on the back and say (for example) "$4f$ minus 20 equals 100" for pupils to work out mentally what f must be.

If pupils seem to be doing too well, they can probably somehow see the number on the back of the card!

Can play in teams.

Answer: 30

1.20.10 Hundred-Square Investigations.
Make a shape (e.g., a rectangle or a letter such as CEFHILOPTUX) out of squares and place it onto a 10 × 10 grid containing the integers from 1 to 100. (See suitable A4 grid for an acetate; in section 1.16 there are smaller ones suitable for photocopying and guillotining for pupils to write on.)
Invent a rule (e.g., "add up all the numbers in the shape"; or "multiply the number in the bottom left square with the number in the top right square"; etc.) and investigate how that value changes when the shape is placed in different positions on the grid.
(It may be convenient to cut out the relevant shape from coloured acetate, so that you can slide it around on an OHP.)

It's best to decide on a position in the shape (e.g., the top left square) as the "reference number" to locate where the shape is placed on the grid.

What if we allow the shape to rotate 90°?

Results can be proved by generalising the situation; e.g., for a U-shape,

x	$x+1$	$x+2$
$x+10$	$x+11$	$x+12$
$x+20$	$x+21$	$x+22$

So if the rule is "add up the left column and subtract it from the total of the right column and multiply the answer by the bottom middle square", then the result will be $(3x+36-(3x+30))(x+21) = 6(x+21)$*;*
i.e., 6 times the bottom middle number.

Alphabet Code

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*m*
9	11	18	2	15	1	20	14	23	6	10	17	26
n	***o***	***p***	***q***	***r***	***s***	***t***	***u***	***v***	***w***	***x***	***y***	***z***
8	19	16	24	4	13	21	3	25	5	22	7	12

Work these out and decode the message.

A

1 $k-5$

2 $3w$

3 $g-u$

4 $6u-1$

5 $m-q$

6 $b+n$

7 $\frac{q}{u}$

8 $5r-5$

B

1 $2a+n$

2 $j+3$

3 $3y$

4 dy

5 $c-w$

6 $2k+3$

7 $w+2r$

8 $2s-v$

9 $\frac{e}{5}$

10 $\frac{m-d}{u}$

C

1 $b+6d$

2 $\frac{t+z+1}{2}$

3 $c+1$

4 $a+d+h$

5 $\frac{30}{d}$

6 $\frac{c}{1+f}$

7 $a+\frac{p}{2}$

8 wr

9 fwu

10 $x-11$

11 $\frac{dr}{2}$

12 $2h-c+1$

D

1 $\frac{j}{d+r}$

2 $m-3$

3 $2r+w$

4 $2y$

5 $\frac{c}{d}$

6 $p-n$

7 $\frac{x}{1+k}$

8 $2f+p$

9 $s+f$

10 $2e-y$

11 $n+\frac{p}{2}$

12 $u+4f+n-2$

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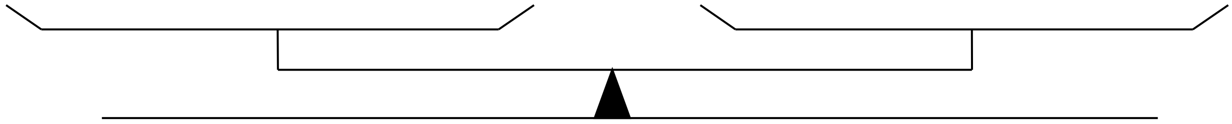

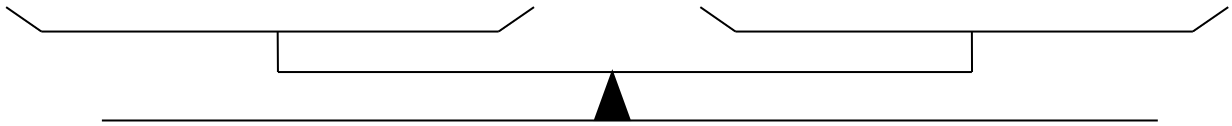

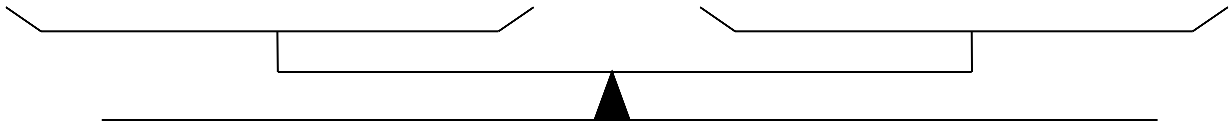

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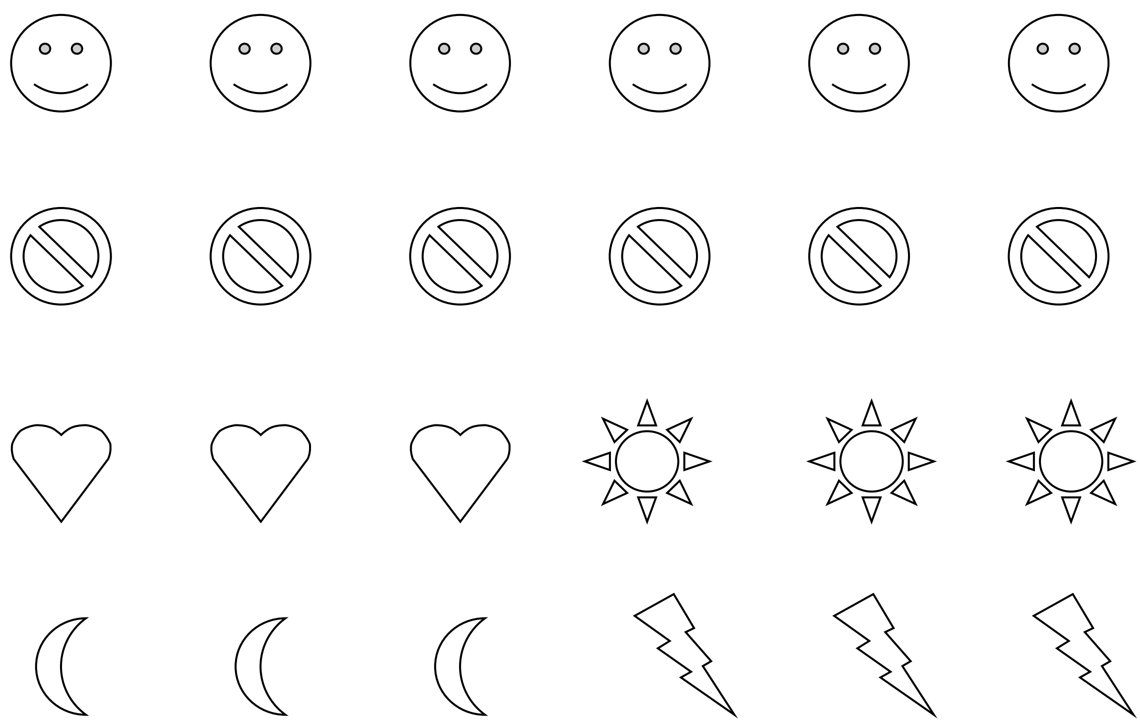
Photocopy onto acetate and then cut out the shapes.
They're easier to use if you leave a gap of 0.5 cm or so around the edge. Keep them in two envelopes.

Shapes to represent unknowns (put in one envelope)



Weights (put in another envelope)

1 kg	**1 kg**	**1 kg**	**2 kg**	**2 kg**
3 kg	**3 kg**	**3 kg**	**2 kg**	**2 kg**
4 kg	**4 kg**	**4 kg**	**5 kg**	**5 kg**
10 kg	**12 kg**	**15 kg**	**5 kg**	**5 kg**
10 kg	**12 kg**	**20 kg**	**6 kg**	**6 kg**

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Photocopy the two sheets back-to-back onto coloured card and then guillotine.
The card must be thick enough for the numbers not to show through.

a	*b*	*c*	*d*
e	*f*	*g*	*h*
i	*j*	*k*	*m*
n	*p*	*q*	*r*
s	*t*	*u*	*v*

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Photocopy the two sheets back-to-back onto coloured card and then guillotine.
The card must be thick enough for the numbers not to show through.

10	3	7	24
13	2	30	1.5
66	9	5	0
73	1	0.1	55
8	3	–2	34

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1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
21	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
31	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
41	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**
51	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**
61	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**
71	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**
81	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**	**90**
91	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**100**

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