1.21 Quadratic Equations

- One way to begin is by "trying" as a class exercise together to solve an equation such as $x^2 + x = 25$ by simple manipulation. Unless pupils see that this is hard, they will wonder why we need some special approach to "quadratic equations". It's only when you try to divide by x, say, and end up with terms like $\frac{25}{x}$ getting in the way, that you realise that it isn't straightforward. So you could just ask for suggestions and do to both sides *exactly* what is suggested.

1.21.1 One way to multiply two binomials is to use boxes. e.g., to expand $(x+3)(x-2)$, write

x	3	
x^2	$3x$	x
$-2x$	-6	-2

So $(x+3)(x-2) = x^2 + 3x - 2x - 6 = x^2 + x - 6$.

So $(x+3)(x-2) = 0$ and $x^2 + x - 6 = 0$ are the same equation.
But the factorised version is easier to solve.

So we need to be able to go backwards from $x^2 + x - 6$, using the boxes, to get $(x+3)(x-2)$.

We write

x		
x^2		x
	-6	

and then try to find numbers to place on each side that multiply to make –6 and have a sum of 1.

It may be worth doing some number work finding pairs of numbers with sums/differences of a certain amount that multiply to give a certain amount, especially involving negative numbers. (See "Number Puzzles" sheet.)
Answers to "Number Puzzles":

question (sum)		***question (difference)***	
1	*2, 8*	*1*	*10, 20*
2	*4, 6*	*2*	*1, 11*
3	*2, 10*	*3*	*2, 10*
4	*1, 11*	*4*	*4, 12*
5	*5, 7*	*5*	*1, 9*
6	*10,10*	*6*	*12, 20*
7	*9, 11*	*7*	*50, 100*
8	*4, 16*	*8*	*30, 50*
9	*6, 10*	*9*	*70. 90*
10	*4, 12*	*10*	*2, 22*
11	*5, 11*		
12	*2, 15*		
13	*5, 12*		
14	*15, 20*		
15	*10, 25*		

You can use instinct or be systematic and write out the pairs of factors of –6 (in this context it's possible to talk about the factors of negative integers): (1, –6); (–1, 6); (2, –3); (–2, 3).

1.21.2 Completing the square or using the formula.
If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, but this formula "goes wrong" if $b^2 < 4ac$ and you don't get any "real" solutions.

You could check answers from previous trial and improvement work – we can now solve those problems much more easily.
If $b^2 > 4ac$, then you get two solutions; if $b^2 = 4ac$ then you get just one solution.

1.21.3 What about when the co-efficient of x^2 isn't equal to 1? If you start with $(rx+s)(tx+u)$ and expand the brackets you get $rtx^2 + (ru + st)x + su$, so a trick for factorising $ax^2 + bx + c$ is to multiply the constant term ($c = su$) by the co-efficient of x^2 ($a = rt$) to get $rstu$. Then you look for two numbers which multiply to make this much and have a sum the same as the co-efficient of x.

For example, to factorise $6x^2 + 11x - 10$, work out $6 \times -10 = -60$ and look for two numbers that have a product of –60 and a sum of 11: they are 15 and –4. So we write $6x^2 + 15x - 4x - 10$, and then factorise this into $3x(2x+5) - 2(2x+5)$ $= (3x-2)(2x+5)$.

This is usually easier than the various other possible methods.

Number Puzzles

We are two numbers.
We add up to 10 and our product is 21.
What are we?   Answer: 3 and 7.

Now try these.

1 We add up to 10 and our product is 16.
2 We add up to 10 and our product is 24.
3 We add up to 12 and our product is 20.
4 We add up to 12 and our product is 11.
5 We add up to 12 and our product is 35.
6 We add up to 20 and our product is 100.
7 We add up to 20 and our product is 99.
8 We add up to 20 and our product is 64.
9 We add up to 16 and our product is 60.
10 We add up to 16 and our product is 48.
11 We add up to 16 and our product is 55.
12 We add up to 17 and our product is 30.
13 We add up to 17 and our product is 60.
14 We add up to 35 and our product is 300.
15 We add up to 35 and our product is 250.

We are two numbers.
This time our *difference* is 10 and our product is 24.
What are we?   Answer: 2 and 12.

Now try these.

1 Our difference is 10 and our product is 200.
2 Our difference is 10 and our product is 11.
3 Our difference is 8 and our product is 20.
4 Our difference is 8 and our product is 48.
5 Our difference is 8 and our product is 9.
6 Our difference is 8 and our product is 240.
7 Our difference is 50 and our product is 5 000.
8 Our difference is 20 and our product is 1 500.
9 Our difference is 20 and our product is 6 300.
10 Our difference is 20 and our product is 44.

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