1.22 Simultaneous Equations

- The main difficulty is with the concept of two (or more) constraints being simultaneously true. Many puzzles involve this type of situation (see below), so beginning with some of these helps with seeing the nature and usefulness of this kind of problem.
- The **substitution** method is simplest when both (we'll assume that there are just two) equations are given as $y =$ something. "If $y =$ *this* and $y =$ *that* then *this* = *that*." "If $a = c$ and $b = c$ then $a = b$".
  The substitution method involves rearranging one equation so that we get an expression for one of the unknowns and then substitution this expression into the other equation.
- **Elimination** is usually more elegant but takes some getting used to.
  Can we really multiply an *equation* (rather than just a *term* or a *side*) by 2, and why doesn't it seem to get twice as big? In fact it's still the "same" equation afterwards. Can we really add two *equations* together? Should we even expect to get an equation as a result? (If we "added" 2 triangles together that wouldn't make another triangle.) Pupils need to see that the 1st equation is telling us that LHS = RHS (they're the same amount), so when we "add up two equations" we're really just adding the same amount to both sides of the 2nd equation: we add the LHS to one side and the RHS to the other, but since LHS = RHS we're really adding the same thing to both sides. When we "multiply an equation by 3", say, we can think of it as "scaling up the scales" – multiplying the weight of everything on both sides of the scales by 3 will still leave it balancing.
- Seeing the connection with intersecting graphs is very helpful, particularly for seeing when either there is no solution (e.g., $x + y = 4$ and $x + y = 5$, parallel lines never cross) or infinitely many pairs of (x, y) values work (e.g., $x + y = 4$ and $2x = 8 - 2y$, equations of the same line).

1.22.1	Can start by trying to "solve" an equation like $x + y = 10$ or an equivalent problem: "I'm thinking of two numbers. When I add them together I get 10. What are the numbers?" To get just one pair of answers, we need an extra condition; e.g., "y is 4 bigger than x" – how can we write that? $y = x + 4$ Substitute the 2nd equation into 1st one and solve. Or draw the 2nd graph and see where it crosses the 1st one.	*There are infinitely many pairs of solutions, not just positive integer but decimal and negative too.* *Pairs of solutions (x, y) all lie on the graph of the equation $x + y = 10$.* *Answer: $x = 3$ and $y = 7$.*
1.22.2	"Fruit machine"–type puzzles (see sheet – suitable for putting on acetate). Pupils can bring in examples of this type of puzzle from newspapers/books/magazines to see if we can solve them more quickly using algebra. *(Sometimes it's faster to use intuition; sometimes it's faster to use algebra.)*	*Answers:* $f = 3$; $h = 4$; $c = 5$; $t = 1$ *(face, heart, etc.);* $s = 4$; $c = 0$; $d = 8$; $h = 2$ *(spade, club, etc.);* $p = 12$; $e = 4$; $y = 5$; $d = 11$ *(pound, euro, etc.);* f *(face)* $= 9$; a *(arrow)* $= 10$; n *(quaver)* $= 4$; m *(2 quavers)* $= 6$; $u = 6$; $d = 3$; $r = 1$; $l = 8$ *(up, down, etc.);* $a = 3$; $b = 1$; $c = 10$; $d = 5$ *(letters).*
1.22.3	Alex has £160 in a bank account and he deposits £8 per week. His sister Becky has £270 in her bank account but she withdraws £3 every week. If they carry on like this, how many weeks will it be before they have the same amount as each other in their bank accounts?	*Let $t =$ number of weeks.* *Then $160 + 8t = 270 - 3t$, so $t = 10$ weeks.* *Or "Becky starts off with £110 more than Alex and at the end of every week Alex ends up £11 better off than Becky (£8 + £3), so it will take 10 weeks before they're equal."*
1.22.4	Circus acrobatics.	*Diagrams make it clear what's going on with the*

A circus family perform acrobatics with the children standing on their parents' shoulders. All the sons are the same height as each other.
The father and his son make a height of 9 ft, and the father and 2 sons make 12 ft. How tall are the man and the sons?

algebra. (Draw a line down the middle of the page and do the algebra on the left and the corresponding drawing – stick people – on the right.)

$f+s=9$ *;* $f+2s=12$ *; so* $f=6$ *ft and* $s=3$ *ft.*

Consider a different family (with different heights). This time if the son stands on the father's shoulders the total height is 7 ft, but if the father holds his son by the ankles, the distance from the ground to the top of the son's head is 3 ft. How tall are this father and his son?

This time, $f+s=7$ *;* $f-s=3$ *; so* $f=5$ *ft and* $s=2$ *ft.*

Pupils can make up their own situations involving acrobats on horseback, stilts, etc.

To make things more complicated, you can introduce a mother (height m) and daughters (height d). Pupils can make up their own puzzles along these lines.

e.g., the mother stands on the father's shoulders holding a son by his ankles who holds his sister by her ankles, etc.

If all four "sizes" of people are involved, how many positions would they have to get into for us to be able to work out their heights with certainty?

For 4 unknowns, you need at least 4 equations.

If they hang over the edge of a drop, then you can have "total heights" which are negative.

1.22.5 Solve these equations simultaneously.

$ab=1$

$bc=2$

$cd=3$

$de=4$

$ea=6$

(Note that none of the equations has a right side of 5.)

Pupils can make up similar sets of complicated-looking simultaneous equations for each other to solve.
Pick what the numbers are going to be first!

There are many methods of solution. One way is to multiply all the equations together to give $(abcde)^2=1\times2\times3\times4\times6=144$*,*
so $abcde=$ *12 or –12.*
Multiplying the 1st and the 3rd gives $abcd=3$*, so* $e=\pm4$*. Therefore* $a=\frac{6}{4}=1\tfrac{1}{2}$*,* $b=\frac{1}{1\frac{1}{2}}=\tfrac{2}{3}$*,*

$c=\frac{2}{\left(\frac{2}{3}\right)}=3$*,* $d=1$ *and* $e=4$*, or all of* a *to* e *could be negative of those values.*

1.22.6 David is twice as old as Henry was when David was as old as Henry is now. If the sum of their ages is 49 years, how old is David?

(Start by working out who is older, Henry or David.)

Answer: 28 years old.
Clearly David is older than Henry, since at some time in the past David was Henry's current age.
There are only 2 unknowns: David's age now (d *) and Henry's age now (* h *), since the time elapsed since David was as old as Henry is now is just* $d-h$ *years.*
Therefore, $d+h=49$ *; and*
$d=2\times(h-(d-h))=2(2h-d)$*, so* $3d=4h$*, so* $h=21$ *and* $d=28$ *years old.*

1.22.7 There are a certain (integer!) number of rabbits and chickens in a cage. If altogether there are 23 heads and 62 legs, how many of each are there?
(Expect to be asked how many legs a chicken has!)

Answer: 8 rabbits, 15 chickens.
You could solve $r+c=23$ *and* $4r+2c=62$ *simultaneously or imagine they were all chickens. That would make 2 × 23 = 46 legs, but there are 62 – 46 = 16 extra legs, and these must come from 8 rabbits. So 15 chickens.*

1.22.8 A mother agrees to pay her daughter £3 every night that she does her homework, provided that the *daughter* pays the *mother* £4 every night that she *doesn't* do any homework.
(Would you agree to this arrangement?!)

Answer: 16 days of homework.
Let h *= number of days that the daughter did her homework. Then she must have skipped her homework on* $28-h$ *days.*

After 28 days, the daughter has paid her mother exactly the same amount of money that her mother paid *her* during that time.
How many days' homework did the daughter do?

Therefore she will have gained $3h-4(28-h)=0$ *pounds altogether.*
Solving this gives the answer.

Alternatively, you can argue that the ratio of days of homework to days of no homework must be $4:3$ *so as to "cancel out" the* $3:4$ *ratio of payment. So we have to split 28 days into the ratio* $4:3$*, which gives* $16:12$*, so she must have done 16 days of homework.*

1.22.9 Helen works as a part-time waitress and gets paid £6 per hour. For overtime she gets £7 per hour. One week she worked for 12 hours and earned £76. How much overtime did she do?

Or assume that it was all at the basic rate. That would have earned her 12 × 6 = £72, but £76 is £4 more so she must have earned those £4 by doing 4 of the hours as overtime.

Answer: 4 hours
This can be tackled in a similar way to above or by letting $x=$ *number of hours of normal work and* $y=$ *number of hours of overtime, so that*

$$6x+7y=76$$
$$x+y=12$$

and then solving these equations simultaneously.

1.22.10 The formula $f=\frac{9}{5}c+32$ converts temperature in Celsius (c) to temperature in Fahrenheit (f). At what temperature are the values the same as each other?

Answer: –40 °C= –40 °F

$$c=\tfrac{9}{5}c+32$$
$$\tfrac{4}{5}c=-32$$
$$c=-40$$

At what temperature is one value twice as big as the other?

Answer: 2 possibilities:
Either $f=2c$*, so*

$$2c=\tfrac{9}{5}c+32$$
$$\tfrac{1}{5}c=32$$
$$c=160$$

and $f=320$*, so 160 ˚C and 320 ˚F;*

or $c=2f$*, so*

$$f=\tfrac{9}{5}(2f)+32$$
$$\tfrac{13}{5}f=-32$$
$$f=-12.31$$

and $c=-24.62$*, so –12.3˚F and –24.6˚C (both to 2 dp).*

Scientists often measure "absolute temperature" in *Kelvin*. The relationship between temperature in Kelvin (k) and temperature in Celsius (c) is approximately $k=c+273$. At what temperature will the values be the same as each other?

Never, because the equations $k=c+273$ and $k=c$ are inconsistent and have no solution.

When will the Kelvin temperature be the same as the Fahrenheit temperature?

If $k=f$, then

$$c+273=\tfrac{9}{5}c+32$$
$$241=\tfrac{4}{5}c$$
$$301.25=c$$

so at about 301 ˚C the Fahrenheit and Kelvin temperatures are both about 574 (K or ˚F).

Pupils can invent similar puzzles to these.

☺	☺	☺	☺	12
☺	♡	☺	♡	14
☺	△	♡	❒	13
♡	△	❒	△	11

♠	♠	♠	♠	16
♣	♠	♣	♠	8
♣	♥	♥	♠	8
♥	♠	♦	♣	14

£	€	¥	¥	26
€	£	£	£	40
$	$	£	£	46
¥	€	€	€	17

☺	☺	↑	♪	32
♫	♫	♫	♪	22
↑	↑	♫	♪	30
☺	♫	☺	↑	34

▲	▲	▼	▼	18
▲	▼	▶	▶	11
◀	◀	▶	▶	18
▲	▲	▲	▼	21

a	*b*	*b*	*c*	15
a	*b*	*c*	*c*	24
d	*a*	*a*	*a*	14
d	*d*	*d*	*d*	20