1.26 Inequalities

- This is a topic which builds heavily on several others: negative numbers and number-lines; solving linear (or other) equations; plotting and interpreting $y = mx + c$ and other graphs. It may be worth reviewing some of these at the outset, depending on what you're aiming to cover.
- It may be quite a jump from saying that if $2x + 3 = 25$ then $x = 11$ to saying that if $2x + 3 > 25$ then $x > 11$, and this can be especially hard to see if negative numbers are involved. It might be helpful to go back to the balancing ideas of solving equations: here we have some unbalanced scales (the heavy left side is down and the right side is up), and if we add the same amount to both sides the imbalance will remain just as it was (and to the same degree).
- One way to approach this topic is to put a hardish inequality on the board (e.g., $12x - 31 > 60$) and ask for any number that will work; e.g., $x = 100$ or $x = 1000000$, and then start looking for the *smallest* number that will work ($\frac{91}{12} = 7.583...$). This can be by trial and improvement initially.
- When shading several graphical inequalities on xy axes, it is often simpler to shade *out* the regions that you *don't* want so that the un-shaded portion left at the end represents the required points. The only problem with this is that the convention of using solid lines for < and > and dashed lines for ≤ and ≥ becomes ambiguous. We have to assume that those conventions apply to the *required* region.

1.26.1 Algebra is not just about equations (when the two sides of the scales balance) but also about when one side is heavier than the other.
There are four inequality signs: $<$, $>$, $\le$ and $\ge$; the first two just more precise ways of saying $\neq$ (not equal to).

You can do "true and false" statements to review these definitions.

A common pupil question is, e.g., "Does $x > 17$ include 17?" which you could answer by asking, "Is 17 more than 17?" No.

These can be quite hard, especially if they include negative or decimal numbers.
"Stand up if true, sit down if false" can be fun.

1.26.2 Number-lines help with seeing that $x < 3$ *and* $x < -2$ (simultaneously true) means that $x < -2$, whereas $x > 3$ *and* $x < -2$ describe no possible values of x.
Also that $x < 3$ *and* $x > -2$ means "between" and can be written as $-2 < x < 3$.

When illustrating inequalities on a number-line, the usual convention is to use a coloured-in *dot to mark the end of an interval if the end is* included, *and an* open *dot if the end-point is* not included.

1.26.3 I'm thinking of an integer (or maybe more than one integer).
1. The number I'm thinking of is less than 40.
2. If I double the number, the answer is more than 55.
3. If I take the number away from 100, I get more than 60.
4. If I multiply the number by 3 and add 4 I get more than 113.

What could my number be?

How many possibilities are there?
Pupils could investigate with a calculator.

Answers: 37, 38, 39

Note that condition 3 duplicates condition 1 (it is redundant).

1.26.4 Are these always true (can you prove it?), sometimes true (tell me when) or never true (tell me why)?
1. $a + b < a - b$
2. $ab < a + b$
3. $ab < \frac{a}{b}$

Answers: ("iff" means "if and only if")
1. *true iff $b < 0$;*
2. *true iff a and b are both < 2;*
3. *true iff $b < -1$ and $a > 0$;*
   or $-1 < b < 0$ and $a < 0$;
   or $0 < b < 1$ and $a > 0$;
   or $b > 1$ and $a < 0$.

1.26.5 True or false?
If $0 < a < 1$, $b > 1$, $-1 < c < 0$ and $d < -1$,
1. $ab < b$

Answers: Pupils can try these with numbers or try to argue through them logically. Not easy.
1. *T*

2. $\frac{b}{a} < b$
3. $\frac{a}{b} < a$
4. $ab < a$
5. $a + b > b$
6. $a + c > a$
7. $a + c > c$
8. $ac > a$
9. $bc < b$
10. $cd > c$

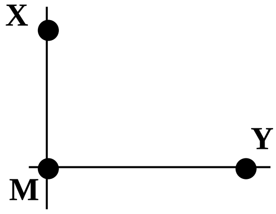
2. F
3. T
4. F
5. T
6. F
7. T
8. F
9. T
10. T

1.26.6 A lift has a mass of 820 kg. If the cable can safely tolerate a maximum load of 1400 kg, how many people (of average mass 70 kg) can it hold?

Answer: 8 people (always round down where safety is concerned!)
$70x + 820 \le 1400$ *gives* $x = 8.29...$

1.26.7 Imagine this situation:
Eighty pupils are lined up in a rectangular array that is 8 rows by 10 columns. I go along every column noting the shortest person in that column. I then pick out the *tallest* of these 10 pupils and give him/her a coloured hat marked with an X.
Next I go along every *row* noting the *tallest* person in that row. Then I pick out the *shortest* of those 8 pupils and give him/her a coloured hat marked with a Y.
Which person is taller, person X or person Y, or can't we be sure?

Answer:
Imagine X and Y in the grid and also pupil M, who is in the same column *as X and same* row *as Y. It doesn't matter how many rows and columns there are in between.*



X is the shortest person in the column, so M must be taller than X ($h_M > h_X$ *).*
(h_M *stands for the height of person M, etc.)*
Y is the tallest person in the row, so Y must be taller than M ($h_Y > h_M$ *).*
Therefore Y is taller than X ($h_Y > h_M > h_X$ *).*

Could person X and person Y be the same person?

Yes, it could happen if there are bigger people in the rest of the column and smaller people in the rest of the row.

1.26.8 Quadratic Inequalities.
The easiest way to solve these is to factorise (or use the equation) so as to write the inequality as something like $(x-a)(x-b) \ge 0$.
Then sketch the graph $y = (x-a)(x-b)$ to see where it is ≥ 0 ; e.g., the solution of $(x-2)(x+3) \ge 0$ is $x \le -3$ or $x \ge 2$ because for these values of x the graph is on or above the x -axis.

It is possible to solve these without drawing graphs, but you have to think very logically.
e.g., if the product of $(x-2)$ *and* $(x+3)$ *is* ≥ 0 *, it means that both must be* ≥ 0 *or both must be* ≤ 0 *, so the possibilities are*
(i) $x-2 \ge 0$ ***and*** $x+3 \ge 0$ *;* ***OR***
(ii) $x-2 \le 0$ ***and*** $x+3 \le 0$.
Now (i) $\Rightarrow x > 2$ *and (ii)* $\Rightarrow x < -3$*, so these are the solutions.*
Where the given inequality is < 0 *or* ≤ 0 *instead, the two brackets must be of* opposite *sign (or zero) and the same method will work.*

1.26.9 Linear Programming.
This can be an interesting application of graphical inequalities.