2.4 Angles

- "Angle" can sometimes refer to a corner (vertex) or to the size of the angle at that vertex.
  In diagrams we tend to use capital letters to represent points or sizes of angles and lower case letters to represent lengths of lines; e.g., A, $\hat{A}$, ABC, $A\hat{B}C$, $\angle A$, $\angle ABC$ for angles and a, AB, for lengths.
- Pupils may also need to know the conventional ways of indicating equal angles, equal lengths of lines and parallel lines in diagrams. This is sometimes better covered when dealing with polygons (section 2.1).
- A full-circle protractor (0°-360°) is a lot more convenient than a semicircle (0°-180°) one, and it's an advantage if there is no area missing in the middle and there are continuous lines going out from the centre to the numbers around the edge. Pupils also sometimes find a 360° protractor easier to hold because of the lump at the centre. The reason for the two scales (clockwise and anticlockwise) may need explaining.
- The convention that anticlockwise rotations are positive and clockwise negative is often used.
- Pupils can aim for an accuracy of $\pm 1°$. A sharp pencil helps. Sometimes lines need extending on a drawing to reach the fine scale around the edge of the protractor.
- Angles are revised in sections 2.5 and 2.6.

2.4.1 People maths. Review N, E, S, W ("Naughty Elephants Squirt Water!", or equivalent). Everybody stand up. Let's say this way is north (or work it out or take a vote! **W**est could be the direction of the **w**indows, etc.).
Which way is clockwise? Which way is right?
Turn 90° clockwise, turn 270° anticlockwise, turn 450° clockwise, etc. Everyone does it at the same time.

Avoid embarrassing pupils who have difficulty distinguishing right and left.
"Right is the hand most people write with."
The first finger and thumb of the left hand make an "L" shape when held out at 90°.

Establish that for 180° the direction doesn't matter.

Then try it mentally. Sit down. If I said turn 270° anticlockwise and then 90° clockwise which way would you be facing? (N, S, E, W?)

If you started off facing North, you would be facing South.

How much clockwise is equivalent to $x°$ anticlockwise?

Answer: $(360 - x)°$, *if* $0 \le x \le 360$, *or more generally* $(360 - x \bmod 360)°$.

Give me directions for getting from here to the hall/dining room/head's office, etc.

Not allowed to draw anything or wave arms around. Imagine you were using the telephone.

2.4.2 **NEED** blindfold (clean tea-towels are convenient), prize (e.g., chocolate bar).
Choose 2 volunteers, one who doesn't mind being blindfolded and blindfold that one. Escort the blindfolded person to the back of the room and rearrange the desks a little. Place the prize somewhere. The blindfolded person has to get the chocolate bar without touching anything except the floor with any part of him/her. The other volunteer has to give directions but isn't allowed anywhere near the blindfolded person.

Keep close by in case the person falls.

Obviously check they can't see anything. Spin round to lose orientation.

On a chair tucked under a table is quite difficult.

You could give them 3 "lives".

Keep it open-ended and see what they do – generally they'll use left and right, but possibly angles, especially near the end when small movements are necessary.

2.4.3 Covering the special names for angles in particular ranges gives a good opportunity to review > and < or to introduce θ (does anyone know Greek?) as a symbol often used to represent an angle (a bit like x standing for a number in algebra).

The words horizontal *(the same direction as the* horizon*) and* vertical *might need revision.*

These may be ways of adding interest to revision material.

- acute angles: $0 < \theta < 90$;
- right angle (quarter turn): $\theta = 90$;
- obtuse angles: $90 < \theta < 180$;
- straight line (half turn): $\theta = 180$;
- reflex angles: $180 < \theta < 360$;
- full turn: $\theta = 360$.

The "angle facts" that the angles on a straight line add up to 180° and the angles at a point add up to 360° are really just definitions of what we count as "straight" and what we mean by "all the way round" together with our choice of how many degrees to have in a full turn.

Why do you think there are 360° in a full turn?

Just a convention/historical accident; apparently the Babylonians counted in 60's (base 60) instead of 10's like we do.
We could use 2π (radians) or 400 (gradients) or anything you like for a full turn.

We could measure in % (25% for a quarter turn, etc.) or "minutes" (15 min for a quarter turn).

2.4.4 Testing Angle Accuracy.
Draw axes from 0 to 10 horizontally and 0 to 30 vertically. Must use the same scale (e.g., 0.5 cm for 1 unit) on both axes.
As accurately as possible, join (0,0) to (10,10). Measure the angle made by this line and the horizontal axis.

Pupils may enjoy the chance to press the mysterious tan *button on the calculator before they learn about it in trigonometry.*

Answer: Should obviously be 45° because we're bisecting the 90° between the axes, but the calculator knows what it should be: make sure you're in degrees mode and do $\tan^{-1} 1$.

Now join (0,0) to (10,20) and again measure the angle this line makes with the horizontal axis.
Finally (0,0) to (10,30).

Check with $\tan^{-1} 2$ *= 63.43494882…°.*

Check with $\tan^{-1} 3$ *= 71.56505118…°.*

Pupils may suspect that the line joining, say, (0,0) to (10,25) would make an angle of $\tan^{-1} 2.5$ *with the* x*-axis.*

Further points may be chosen if more practice is needed; this saves photocopying sheets of random angles for pupils to measure, and reviews co-ordinates at the same time. It may also be more interesting.

2.4.5 What is an angle?
(Imagine your little brother/sister wanted to know what your maths homework was about.)

It's quite hard to explain; you can't really say "distance between", etc.
Need to talk about rotating or turning: "a way of saying how much something has turned".

Tell me a job in which you'd have to think about angles?

flying a plane, footballer, plumber, roofer, TV aerial fitter, maths teacher, etc.

2.4.6 Parallel and perpendicular logically crop up here, because parallel lines are straight lines going in the same direction (angle between them = 0°) and perpendicular lines are lines that are at 90° to each other. (Even if the lines go on for ever in both directions, perpendicular lines don't necessarily touch in 3 dimensions. Non-parallel non-intersecting lines are called *skew* lines.)

It isn't enough to say that parallel lines are just lines the same distance apart because so are the circumferences of concentric circles or railway tracks going round a bend. The lines also have to be straight.

Could discuss infinity. Do parallel lines ever meet? Not if they're always a certain (non-zero) distance apart. On a globe "parallel" lines of longitude meet at the N and S poles.

Pupils may know from reflection and refraction in science that "normal" means at 90° (the angles of light rays are measured from lines normal to the surfaces).

2.4.7 Find out how much the Leaning Tower of Pisa leans (could do this for homework).

Answer: about 10°, although it varies year by year as it leans more and then engineers try to straighten it a little.

2.4.8 **NEED** "Which Angles are Equal?" sheets, pencil crayons, protractors perhaps.

The intention is that pupils draw a circle (about 1 cm radius) at each of the 15 crossing points and then using two colours shade vertically opposite angles the same colour. Equal angles at different crossing

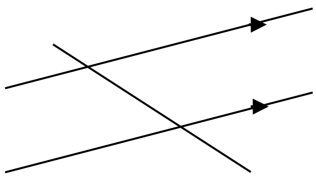
This seems to work better than asking pupils to draw *their own lines, because unless you're careful there isn't room to mark the angles clearly. Have some spare copies because when you've gone very wrong it's hard to rescue!*

points should also be coloured the same colour.

How many colours will you need?

Answer: 6, because there are 3 pairs of parallel lines, and each can intersect any of the others – and $^3C_2 = 3$. Each intersection creates 2 different-sized angles, so altogether there will be 6 different angles.

2.4.9 Angles associated with parallel lines.
Learning the names can be tedious.
First concentrate on identifying which angles are equal to each other and which pairs of angles sum to 180° (supplementary angles).
If you have a rectangular whiteboard and a metre stick (or pole for opening the windows) you can easily lay the stick across parallel sides of the board at different angles to show what equals what; otherwise use a noticeboard or desk.



The line crossing a pair of parallel lines is called a transversal.

Vertically opposite *angles (like scissors or a pair of pupils' rulers, like letter X) – nothing to do with a "vertical" direction, but angles opposite at a "vertex" (= point).*
Corresponding *angles (in corresponding positions, like Chris and Katie, both at the end of a row in the classroom, like letter F).*
Alternate *angles (opposite sides of the line that goes through the parallel pair of lines, like letter Z).*

Interior *angles – the odd ones out because they're not equal but they sum to 180° (like letter C).*

(Unfortunately "C-angles" are "interior" and the "C" doesn't stand for "corresponding".)

Interior angles in polygons *are just the inside angle at each vertex, and they sum to different amounts depending on how many sides the polygon has.*

2.4.10 Clock Angles.
If you look at the minute and hour hands of a clock, there are two angles between them.
I'm only interested in the smaller of the two angles.
Tell me a time when the angle is 90°.
Why is 12.15 not exactly correct?
Will 12.15 be more or less than 90°?
Tell me another time when the angle will be just less than 90°?
When will it be just more?
What will the angle be at 9.30?

Answers:
This smaller angle may be acute, right-angled or obtuse but never reflex.

3.00 and 9.00. Other times are probably not exact.
The hour hand will have moved a bit as well.
Less.
Less: 1.20, 2.25, 3.30, 4.35, 5.40, 6.45, etc.

More: 4.05, 5.10, 6.15, 7.20, 8.25, 9.30, etc.
Hour hand will be half way between 9 and 10 (15° above the 9) so the angle is 90 + 15 = 105°.

Work out the (smaller) angle between the hands at

1. 3.00	**2.** 11.00	**3.** 11.30
4. 3.30	**5.** 12.05	**6.** 4.45

(Pupils can do some of this mentally.)

Answers:

1. 90°	**2.** 30°	**3.** 165°
4. 75°	**5.** 27.5°	**6.** 127.5°

Make use of symmetry (3.30 will be the same angle as 8.30 – not 9.30 – because it's just a reflection in a mirror).

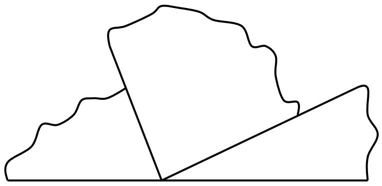
2.4.11 The sum of the three interior angles in a triangle = 180°.
NEED small paper triangles, one per pupil.
Hand them out. Who's got a nearly right-angled triangle? Who's got a nearly equilateral triangle?, etc. Hold it up.
Class experiment – between us we're trying all sorts of triangles.

*Be sure that pupils don't think you're saying "***some** *of the angles in a triangle are 180° – and some aren't"!*

You can make these quickly and easily using two or three sheets of coloured A4 paper and a guillotine. Make sure that there are a variety of acute-angled, obtuse-angled and right-angled triangles.

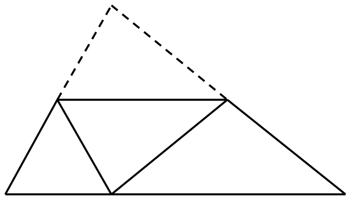
Option 1: You can tear off the corners and arrange the pieces next to one another to make a straight line.

Tearing is better than cutting because it's easy to see which corner is the angle *that was previously in the triangle.*



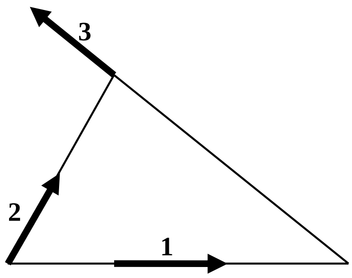
Option 2: Put the longest side horizontal (largest angle at the top) if you can tell, and measure the lengths of the two shorter sides; divide those lengths by two to find the mid-points of those two shorter sides and mark them. Join them with a ruler to give a line parallel to the bottom side and fold the top part down along this line (see diagram right).
Fold in the other two corners (vertical fold lines) and all three angles should meet together and make 180°.

Option 2 may be better if many pupils have done option 1 before.



Note that neither option 1 nor option 2 is a proof, *although they can be turned into proofs by thinking about equal angles.*

Dynamic Proof: Draw a “general” triangle on the board and put a large cardboard arrow or similar straight object (must have distinguishable ends) in the middle of the bottom side (see diagram right, **1**). We’re going to use the arrow to “add up” angles. Slide it to the left corner (no change in direction so no angle yet) and rotate it to “measure” the interior angle at that corner (**2**). Slide it to the top corner, and, still rotating anticlockwise, “add on” that corner angle (actually the angle vertically opposite to it, which is equal) (**3**). Slide it, without changing direction, to the final (right) corner and “add on” that angle (still anticlockwise), leaving the arrow pointing 180° relative to its initial position (**1**).



(This is similar to the standard proof that the exterior angles of any polygon add up to 360° by walking around it.)

Work like this shows the dynamic aspect of angles – they’re a measure of turning movement.

2.4.12 How many times in the course of 12 hours are the hands of a clock at right angles to each other?

Pupils can invent similar puzzles.

Answer: 22 times
Imagine starting and finishing at 12.00.
The hands will be at right angles twice in every one hour period, except that we will count 3.00 and 9.00 twice (because they occur on the hour), so we have to subtract those two.
24 – 2 = 22 times.

2.4.13 The hour hand and minute hand on an analogue clock coincide at 12 noon. When is the next time when they coincide exactly?

Hint: It won’t be exactly 1.05 pm.
Will it be before or after? *(After)*

If they coincide t *hours after 1 o’clock, then* $\theta_h = 30(t+1)$ *and* $\theta_m = 360t$*, so solving simultaneously gives* $30(t+1) = 360t$ *and so* $t = \frac{1}{11}$ *hour after 1 o’clock.*

Answer: Can use simultaneous equations, even the concept of angular velocity, if you like.

A neater way is to see first that the occasions when the hands coincide will occur regularly. (At the 1.05-ish time, imagine rotating the painted numbers so that it reads 12 noon again – you could carry on like this.)
Since it will happen 11 times in 12 hours, each coincidence will occur after $\frac{12}{11}$ *of an hour; i.e., at 1.05 and 27 secs, 2.10 and 55 secs, etc.*

2.4.14 Triangles. Draw up a table like this (big enough to contain *drawings*):

	scalene	**isosceles**	**equilateral**
acute-angled			
obtuse-			

There are two systems for naming triangles: by their angles or by the lengths of their sides.

Note that “acute-angled” means all the angles are acute, whereas “obtuse-angled” means only that there is one obtuse angle (more would be impossible

angled			
right-angled			

For the top left square, if a triangle can be both scalene and acute-angled, draw an example. Put X if it's impossible, and try to say why. Complete the table.

– see section 2.4.15).

Answers:
Equilateral triangles must be acute-angled, because all the angles have to be 60°.
All other combinations are possible.

2.4.15 What sorts of angles can a triangle have?

Here are some more specific questions.
If the answer is yes, make up an example and draw or list the angles; if the answer is no, try to explain why not.

Can a triangle have (as *interior* angles) …

1. an obtuse angle?
2. two obtuse angles?
3. an obtuse angle and a right angle?
4. an obtuse angle and an acute angle?
5. a reflex angle;
6. two right angles;
7. a right angle and an acute angle?
8. four acute angles?

Answers:

1. *yes, lots of examples;*
2. *no, because their total would be >180°;*
3. *no, because their total would be >180°;*
4. *yes, lots of examples;*
5. *no, because it would be >180°;*
6. *no, because their total would be 180° and there needs to be a non-zero third angle;*
7. *yes, lots of examples;*
8. *no, because although their total might be 180° (e.g., 40˚, 45˚, 45˚, 50˚), a "tri-angle" has to have exactly 3 angles!*

Note that we are always talking about the interior *angles of the triangles.*

2.4.16 I'm thinking of a triangle. Tell me how big its angles are and what kind of triangle it is.
I'll call the angles in each one A, B and C.

1. Angle B is twice the size of angle A and angle C is three times the size of angle A;
2. Angle A is twice the size of angle B and angle C is the same size as angle B;
3. Angle B is twice the size of angle A and angle C is three times the size of angle **B** (not same as question 1);
4. Angle B is four times the size of angle A and angle C is five times the size of angle A.

Answers:

1. *A = 30°, B = 60°, C = 90°; scalene right-angled triangle;*
2. *A = 90°, B = 45°, C = 45°; isosceles right-angled triangle;*
3. *A = 20°, B = 40°, C = 120°; scalene obtuse-angled triangle;*
4. *A = 18°, B = 72°, C = 90°; scalene right-angled triangle.*

These can be solved by forming equations or by trial and improvement or by "intuition".

Pupils can make up their own for each other.

2.4.17 Polygon Angles.
("Interior" angles in polygons means something different from "interior" angles formed when a straight line crosses a pair of parallel lines.)

Drawing polygons with various numbers of sides and measuring and summing the interior angles of each tends to give very inaccurate results, although it is a possible approach.

What do the angles inside a square add up to?
What if I just draw any quadrilateral?
(Keep it convex for now.)

Most will know/guess 360°.
Show how it can be split by a diagonal into two triangles. Colour the angles in one triangle red and the other blue. What do the blue angles add up to?, etc.

Use this trick of splitting into triangles to work out the total interior angle in polygons with sides from 5 all the way up to 10. Choose one vertex and joining this to all the other vertices, so dividing the polygon into triangles.
If the polygon happens to be regular, what can you say about each of the interior angles?

If the polygon has n sides (and so n vertices), this method will divide it into $n-2$ triangles.
So the total interior angle = $180(n-2)$.

In a regular polygon, the interior angles will be equal, so each will be $\frac{180(n-2)}{n} = 180 - \frac{360}{n}$.

If the polygon contains one or more reflex angles,

this method of dissection into triangles doesn't work. In that case, one or more points have to be chosen inside the triangle and these joined to as many vertices as possible. In this way, the polygon can always be divided into triangles, and each additional internal "point" contributes an extra 360° to the total angle, and this has to be subtracted.
(The total interior angle for a concave polygon like this is always the same as for a convex polygon with the same number of sides.)

no. of sides n	***no. of triangles*** $n-2$	***total interior angle***	***size of each angle if polygon is regular***
3	*1*	*180*	*60*
4	*2*	*360*	*90*
5	*3*	*540*	*108*
6	*4*	*720*	*120*
7	*5*	*900*	*128.6*
8	*6*	*1080*	*135*
9	*7*	*1260*	*140*
10	*8*	*1440*	*144*

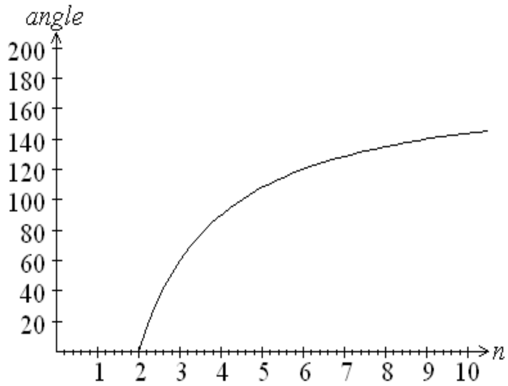
Plot a graph of "size of one interior angle of a regular n -gon" against n.
Make a prediction for $n = 100$.
Why does this happen?

The graph gets closer and closer to 180° as the number of sides the polygon has goes up. (There is an asymptote at 180˚.)
When $n = 100$*, angle = 176.4°.*
As the number of sides goes up, the polygon looks more and more like a circle, and if you zoom in on any part of the circumference of a circle it looks almost like a straight line (180° interior angle).

Mathematically, if θ *is the angle, then*

$$\theta = \frac{180(n-2)}{n} = 180 - \frac{360}{n}$$

, and as n *gets larger and larger,* $\frac{360}{n}$ *gets smaller and smaller, so that* θ *increases, getting closer and closer to 180°.*
(As $n \to \infty$, $\theta \to 0$.)

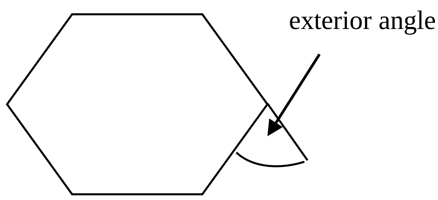


Common-sense says the angle at each corner will get bigger as the polygon gets more sides.

An alternative way to find the interior angle of a regular polygon is to begin with the exterior *angles (which sum to 360° for any polygon), so in a regular polygon each must be* $\frac{360}{n}$.
Therefore, since each exterior angle and interior angle together must make a straight line, each interior angle must be equal to

$$180 - \frac{360}{n} = \frac{180(n-2)}{n}.$$

2.4.18 Exterior angles.
NEED newspaper and scissors.
The "exterior angle" doesn't mean all the angle outside the shape at each vertex; it means the *change in direction* at each vertex, so that exterior angle + interior angle at each vertex equals 180° (see diagram right).
(For a concave polygon, where the interior angle at one or more vertices is a reflex angle, we say that there is no exterior angle at that vertex.)
Place sheets of newspaper on the floor and imagine walking round it (start in the middle of one of the sides). The total change in direction is 4 × 90° = 360°. This will always happen. (Cut the paper so that it makes a more unusual polygon and do it again.)
After we've walked round the whole shape we're always pointing back in the same direction.



(Or you can go outdoors and use or draw with chalk a shape on the ground.)

For this to work with a concave polygon we need to count the exterior angle where there is a reflex interior angle as negative*, because at that corner we change from going clockwise to going anticlockwise (or vice versa).*

With an equilateral triangle, the exterior angles are 120°. Exterior angles are important when doing LOGO programming (see section 3.8).

2.4.19 Any convex polygon with n sides can be split up into $n-2$ triangles. How many ways are there of doing that?

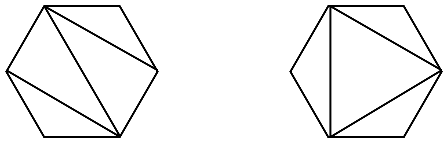
Answer:

n	***no. of ways***	n	***no. of ways***
3	*1*	*8*	*132*
4	*2*	*9*	*429*

They fit the formula

number of ways = $\dfrac{{}^{2n-4}C_{n-2}}{n-1}$.

For instance, for $n=6$ *you can have, e.g.,*



5	*5*	*10*	*1430*
6	*14*	*11*	*4862*
7	*42*	*12*	*16796*

These are the Catalan *numbers (Eugène Catalan, 1814-1894).*

It might seem that for $n \geq 5$ *there will just be* n *ways, because* $n-3$ *vertices can be joined to each of the* n *vertices, but there are more than this because the cuts need not all meet at one point. For* $n \geq 6$ *there are dissections like those on the left.*

2.4.20 Constructing and solving equations from polygon and parallel line angles;
e.g., a triangle has angles $2x$, $3x$ and $5x$; how much are they?

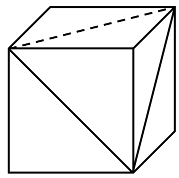
Answer: $10x = 180$*, so* $x = 18°$*, so the three angles are 36°, 54° and 90°.*

It's easy to make up this kind of thing or find examples in books.

2.4.21 Angles in 3 dimensions.
Two diagonals are drawn on two adjacent faces of a cube. If they meet at a vertex, what angle do they make with each other?

People often go for complicated methods like trigonometry or vectors, but particularly in geometry there is often an easier and more elegant solution.

Answer: need to do a diagram or make a model.



If you add in a third diagonal, you can see that you get an equilateral triangle (all sides are the same length), so the interior angles are all 60°.

2.4.22 How much can an object lean without toppling? For instance, a 3 × 1 × 1 cuboid brick standing on its end – what's the steepest slope it can balance on?
For this brick,
maximum angle of slope = $\tan^{-1}\frac{1}{3} = 18.4°$.

Answer: If the object is uniform (the same all the way through), then its "centre of mass" will lie at the geometric centre. The object will be stable if a vertical line going through the centre of mass passes within the base.
(We assume that there is plenty of friction so the block won't slide *down the slope.)*

Which Angles are Equal?

- Use arrows to show which lines below are parallel to each other.
- Every time two lines cross each other, they create four angles.
  There are 15 crossing points in the drawing below, so there are 60 angles.
  Use colour to show which angles are the same size as each other.