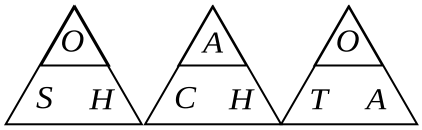
2.6 Trigonometry

- A topic that builds heavily on many others: ratio, similarity and enlargement, angles and lengths, calculator use, rearranging formulas, Pythagoras' Theorem, rounding to dp or sf.
- SOHCAHTOA is a common and not too difficult mnemonic. But some people prefer a whole sentence; e.g., "*S*everal *O*ld *H*orses *C*art *A*way *H*appily *T*onnes *O*f *A*pples." Pupils may remember them better if they invent their own and make them funny.
  A convenient way for using SOHCAHTOA is as three formula triangles (below); these can be written at the top of a page of trigonometry work. Each formula triangle has one of A, O or H missing, so we ask "What *aren't* we interested in?" Let's say we don't know the adjacent side and we don't need to work it out, so we want the formula without A in it; therefore $\sin$, the first triangle formula.
  Then decide whether we need the opposite side, the hypotenuse or the angle, and write down a formula for opp, hyp or $\sin x$ from the first triangle formula.



  These work by covering over the variable you want to reveal the formula for it.
  (If you cover S, you can see $\sin x = \frac{opp}{hyp}$; cover H and you can see $hyp = \frac{opp}{\sin x}$, and cover O and you can see S and H next to each other, so $opp = \sin x \times hyp$. Similarly for the other formula triangles.)
- Pupils can get into a habit of putting all the information onto a clear drawing and labelling the three sides with hyp, opp and adj before they start any calculations.
- Remind pupils to have calculators in the correct mode (degrees or radians). Sometimes the simplest way to sort out a messed-up calculator is to use a biro to press the "reset" button on the back.

2.6.1 **NEED** "Trigonometry Investigation" sheets.
Clearly all the triangles are similar (enlargements of one another), and any ratio of corresponding sides like this should be equal.

You can also measure hyp and calculate the $\sin$ and $\cos$ ratios.
($\sin 35 = 0.57$ and $\cos 35 = 0.82$, each to 2 dp.)

This task takes advantage of the fact that $\tan 35$ *is very close to 0.7 (actually 0.7002075…).*

Approximate results:

	***opp* (mm)**	***adj* (mm)**	***opp/adj* (2 dp)**
a	*35*	*50*	*0.70*
b	*63*	*90*	*0.70*
c	*73*	*105*	*0.70*
d	*46*	*65*	*0.71*
e	*32*	*45*	*0.71*

2.6.2 You could begin the topic by investigating the "mysterious" $\sin$, $\cos$ and $\tan$ buttons on the calculator.
They are functions that convert an angle in degrees (make sure they're in degrees mode) into a number.
What's the biggest number you can make $\sin$ give you?
What's the smallest?
What's it got to do with a right-angled triangle that has that size angle as one of its angles?

1, –1 ($\sin$ *values are always between 1 and –1).*

Pupils can experiment with the functions to see what happens.

2.6.3 Plot graphs of $\sin x$, $\cos x$ and $\tan x$ against x and discuss their properties.

Things to spot include these:

- $\sin x$ *and* $\cos x$ *are both sinusoidal in shape and periodic functions with the same period of 360°;*

Plot values for $0° \le \theta \le 720°$, say.
What do you notice?

$\cos x$ is just 90° ahead of $\sin x$ (called a "phase shift").

- *They both take values between – 1 and 1 ("amplitude" = 1).*
- *$\tan x$ is different in having a period of only 180° and in taking all values from $-\infty$ to ∞. It is also a discontinuous graph, with vertical asymptotes at $\pm 90°$, $\pm 270°$, etc.*
- *$\cos x$ is an* even *function, symmetrical about the vertical axis ($\cos(-x) = \cos x$), whereas $\sin x$ and $\tan x$ are* odd *functions, having rotational symmetry of order 2 about the origin ($\sin(-x) = -\sin x$ and $\tan(-x) = -\tan x$).*

Key values of the functions at 0°, 30°, 45°, 90°, etc. can be noted (see sheet).

You could discuss the fact that sinusoidal graphs turn up in science; e.g., alternating current/voltage; electric/magnetic fields in electromagnetic waves (e.g., light); the sound wave of a tuning fork is nearly sinusoidal.

Fourier Analysis *(Jean Baptiste Fourier, 1768-1830) is a way of building up a waveform by adding together sinusoidal waves of different amplitudes and frequencies.*

2.6.4 Snell's Law in Science
(Willebrord Snell, 1580-1626).
When a beam of light travels from one medium to another, the angle of refraction r (in the second medium) is related to the angle of incidence i (in the first medium) by Snell's Law, $\sin r = \frac{\sin i}{n}$, where the constant of proportionality n is the refractive index.

Plot r against i for a couple of different values of n; e.g., $n = 1.5$ (for light going from air into glass) and $n = \frac{1}{1.5} = \frac{2}{3}$ (for light going from glass into air).

For air and water the values are ${}_{air}n_{water} = \frac{4}{3}$ (air to water) and ${}_{water}n_{air} = \frac{3}{4}$.

Answers:
Angles are measured from the normal (a line perpendicular to the boundary).

For $n = 1.5$, r increases less quickly than i, and when i reaches its maximum value of 90°, $r = \sin^{-1}(\frac{1}{n}) = 41.8°$, the so-called "critical angle" for glass-air.
For $n = \frac{2}{3}$, r increases more quickly than i, and this time when i reaches the critical value of 41.8°, i no longer has any value (because $\sin$ can never be >1). This corresponds to the light beam not being refracted at all: "total internal reflection" takes place.

For water-air, the critical angle is 48.6°.

2.6.5 Converting gradients into angles.
We already know that the line $y = x$ (gradient of 1) makes an angle of 45° with the x-axis. What is the angle that the line $y = 2x$ makes?

Answers:
$y = 2x$ makes an angle of $\tan^{-1} 2$ = 63.4°.
$y = mx$ makes an angle of $\tan^{-1} m$, and if you count a negative angle as a clockwise turn, then this works even when m is negative.

2.6.6 **NEED** Old-fashioned trigonometrical tables.
You could explain how hard life was in the old days (!) or how clever calculators are today being able to work out $\sin$, $\cos$ and $\tan$ so quickly. Pupils could try to find out for homework how the calculator does it.

Pupils can try this out for themselves on a spreadsheet. Can you beat the calculator by finding the next decimal place after the last one the calculator shows?

Answer:
They use power series such as
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ *in radians.*
Taking enough terms, you can get an answer as accurately as you like. (You first have to convert the angle from degrees.)

Spreadsheets can handle more decimal places than most calculators.

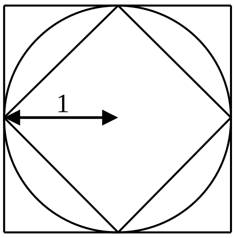
2.6.7 Assuming that the Leaning Tower of Pisa leans at an angle of 10° and is 55 m high, how far does the top lean out over the bottom?

Answer: $55 \tan 10 = 9.7$ *m.*

2.6.8 Approximating π from regular polygons.
Two approaches (see below and right):
1. Area:
π is the area of a unit circle (a circle of radius 1) and

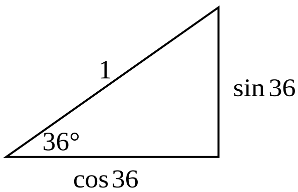
2. Perimeter and Circumference:
2π is the circumference of a unit circle, so we can

we can approximate that area by finding regular polygons that just fit into the unit circle and just fit outside the unit circle and working out their areas. Start with squares.

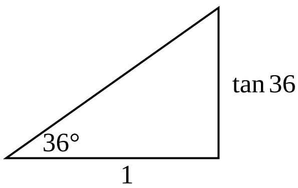


The outer square has area 2 × 2 = 4 sq units.
The inner square has area $4 \times \frac{1}{2} \times 1 \times 1 = 2$ sq units (by dividing it up into four triangles).
So this gives $2 < \pi < 4$ (not a very good approximation, but it's a start!).

Next try regular pentagons, and so on.
The pentagons can be divided into 5 congruent isosceles triangles by joining each vertex to the centre of the circle. Each of these triangles can be cut in half to give two congruent right-angled triangles.
For the pentagon *inside* the circle, we have



so total area = $10 \times \frac{1}{2} \cos 36 \sin 36 = 5 \cos 36 \sin 36$.
For the pentagon *outside* the circle, we have



so total area = $10 \times \frac{1}{2} \tan 36 = 5 \tan 36$.
So now $5 \sin 36 \cos 36 < \pi < 5 \tan 36$, giving 2.38< π <3.63, a better approximation.

In general, for an n-sided polygon we obtain $n \sin\left(\frac{180}{n}\right) \cos\left(\frac{180}{n}\right) < \pi < n \tan\left(\frac{180}{n}\right)$, so putting in n = 100 gives 3.1395< π <3.1426, and taking n = 100 000 gives π as accurately as can be displayed on a calculator.

2.6.9 Rabbit Run. An investigation involving similar work to the above approximating π methods.
I want to make a rabbit run area in my garden and I want to make the area as large as possible. (My garden is very big.) I have 24 metres of plastic fencing and I want to know what shape run will enclose the maximum possible area.

approximate that length by finding the perimeters of regular polygons that just fit into and outside the unit circle; i.e., the same regular polygons as used on the left.

Here, the outer square has perimeter 4 × 2 = 8 units and the inner square has perimeter 4 × $\sqrt{2}$ = 5.66 units, using Pythagoras' Theorem.
So $5.66 < 2\pi < 8$, so $2.83 < \pi < 4$, giving a different (slightly better) approximation from that on the left.

Using the same drawing as on the left, we have total perimeter of the pentagon *inside* the circle = $5 \times 2 \sin 36 = 10 \sin 36$.

And total perimeter of the pentagon *outside* the circle = $5 \times 2 \tan 36 = 10 \tan 36$.
This gives $10 \sin 36 < 2\pi < 10 \tan 36$ or $5 \sin 36 < \pi < 5 \tan 36$, giving 2.94< π <3.63.

In general, for an n-sided polygon we obtain $n \sin\left(\frac{180}{n}\right) < \pi < n \tan\left(\frac{180}{n}\right)$, so putting n = 100 gives 3.1411< π <3.1426.

Using perimeters of regular polygons gives a narrower approximation for π for a given value of n than using areas.

This method is interesting, but using a calculator to find values of sin , cos *and* tan *isn't really valid because the calculator uses its value of* π *to work out them out!*
The rectangular case is an investigation in section 2.2.7.
As shown there, there are 6 rectangles with integer sides and perimeter 24 m.

rectangle	*area*	*rectangle*	*area*
1 × 11	*11*	*2 × 10*	*20*
3 × 9	*27*	*4 × 8*	*32*

Start with rectangular runs.

5 × 7	*35*	*6 × 6*	*36*

The one which encloses the maximum area is the square one (6 × 6).

What if you set up the run (still rectangular) against the garden fence (then the fencing has to go round only three sides)?

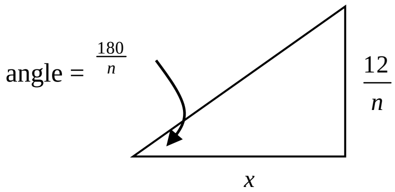
Answer: best option is a 6 × 12 run.
(The function $A = x(24-2x)$ *has its maximum value when* $x = 6$*; could draw a graph.)*

What if you use the *corner* of the garden, so that the run (still rectangular) only has to have fencing on *two* of the sides?

Answer: best option this time is a 12 × 12 run.
(The function $A = (12+x)(12-x)$ *has its maximum value when* $x = 0$*.)*

What if you are allowed to make any shape run you like. Which will enclose maximum area?
Start in the middle of the garden (away from the garden fence) and find the best shape. Then try it next to the fence.

For an n*-sided regular polygon in the middle of the garden, we can put a point in the centre and join it to each of the vertices to obtain n congruent isosceles triangles. We can divide each of these into two congruent right-angled triangles (below).*



We can work out that $x = \dfrac{\frac{12}{n}}{\tan\frac{180}{n}} = \dfrac{12}{n\tan\frac{180}{n}}$*, so the total area of the polygon (*$2n$ *of these right-angled triangles) is* $2n \times \frac{1}{2}x\frac{12}{n} = 12x$

$= \dfrac{12\times12}{n\tan\frac{180}{n}} = \dfrac{144}{n\tan\frac{180}{n}}$.

As n *gets larger and larger, this area gets closer and closer to the area of a circle with circumference of 24 m.*
Radius = $\frac{24}{2\pi} = 3.82$ *m, so area =* $r^2\pi = 45.84$ *m²*.
The ***isoperimetric theorem*** *says that a circle encloses the maximum possible area for a given perimeter. (This is why soap bubbles are spherical.)*

Using the garden fence as just one of the sides, a semicircle would be best. The curved portion would have length 24 m, so the run would be half of a circle of circumference 48 m, so the radius would be $\frac{48}{2\pi}$ *= 7.64 m, and the area would be* $\frac{1}{2}r^2\pi = 91.67$ *m²*.
By using the corner of the garden you could do even better. The run would be a quadrant, the radius would be $\frac{4\times24}{2\pi}$ *= 15.28 m and the area would be* $\frac{1}{4}r^2\pi = 183.35$ *m²*.

2.6.10 **NEED** various equipment. Trigonome-tree!
Calculate the height of a tree in the school grounds (or nearby) using a clinometer
(a device for measuring angle of elevation – you can make a reasonable one by sticking a protractor onto a piece of cardboard and using a ruler as the rotating portion).

Could estimate the volume of the tree and perhaps estimate how many paper towels (from school toilets) could be made from one tree.

Choose a reasonably tall and thick tree. Assume it is cylindrical (or conical) and measure the circumference with a tape measure.

2.6.11 True or false? Try them with angles.
1. $\sin(x+y) = \sin x + \sin y$
2. $\sin 2x = 2\sin x$
3. $\sin 2x = 2\sin x \cos x$
4. $\sin(90-x) = \sin x$
5. $\sin(180-x) = \sin x$

… and similar with $\cos$ and $\tan$.
Make up some more like these.
Try to prove those that are true by using graphs of

Answers:
1. *False unless* x *or* $y = 0°$*, 360°, etc.;*
2. *False unless* $x = 0°$*, 360°, etc.;*
3. *Always true;*
4. *False unless* $x = 45°$*, 225°, etc.;*
5. *Always true.*

the functions.

2.6.12 Trigonometry Facts and Formulas (see sheets).

2.6.13 Methane (CH_4) is a tetrahedral molecule.
The carbon atom is surrounded by four hydrogen atoms that get as far away from each other as they can while staying the same distance from the carbon atom.
Calculate the H-C-H bond angle.

Could use molecular models (science dept.).

You could use vectors to solve this problem.

Answer: 109.5° (see sheet)
(This is a very well-known value among chemists, although few could calculate it geometrically!)

You can imagine the hydrogens on the surface of a sphere with its centre at the carbon.
If one or more of the H's are replaced by a different atom, the bond angles won't all be exactly equal any more (unless all four H's are replaced; e.g., in tetrachloromethane, CCl_4).

2.6.14 Find out in which Sherlock Holmes story the detective has to use trigonometry to solve the mystery.

Answer: "The Adventure of The Musgrave Ritual" from "The Memoirs of Sherlock Holmes" by Arthur Conan Doyle.

He really just uses similar triangles. "If a rod of 6 feet threw a shadow of 9 feet, a tree of 64 feet would throw one of 96 feet." This enables him to locate some hidden treasure and solve the case.

2.6.15 Cinema/Theatre Seats.
What is the optimum slope of the floor in a cinema or theatre so that everyone can see over the heads of the people in front?

You could discuss the way that these sorts of calculations using "averages" across the population can fail to satisfy anyone: seats designed for an "average" person may be uncomfortable for almost everyone.
You could estimate how awkward it might be for someone who is very tall or very short.
Who would be worse off?

Of course, this won't work if the person in front is wearing a hat!

Answer:

$$\tan^{-1}\frac{\textbf{average eye-to-top-of-head distance}}{\textbf{average thigh-to-knee distance}}$$

If you imagine someone sitting directly behind another person, their eyes need to be above the top of the head of the person in front.
The seat needs to go back at least as far as the distance from their knee to the top of their leg.

Putting in estimates for these gives an angle of about $\tan^{-1}\frac{15}{60} = 14^{\circ}$ *approximately.*

Estimate the maximum slope that you think would be safe, bearing in mind that the floor will get wet when it's raining outside.
What about someone using a wheelchair?

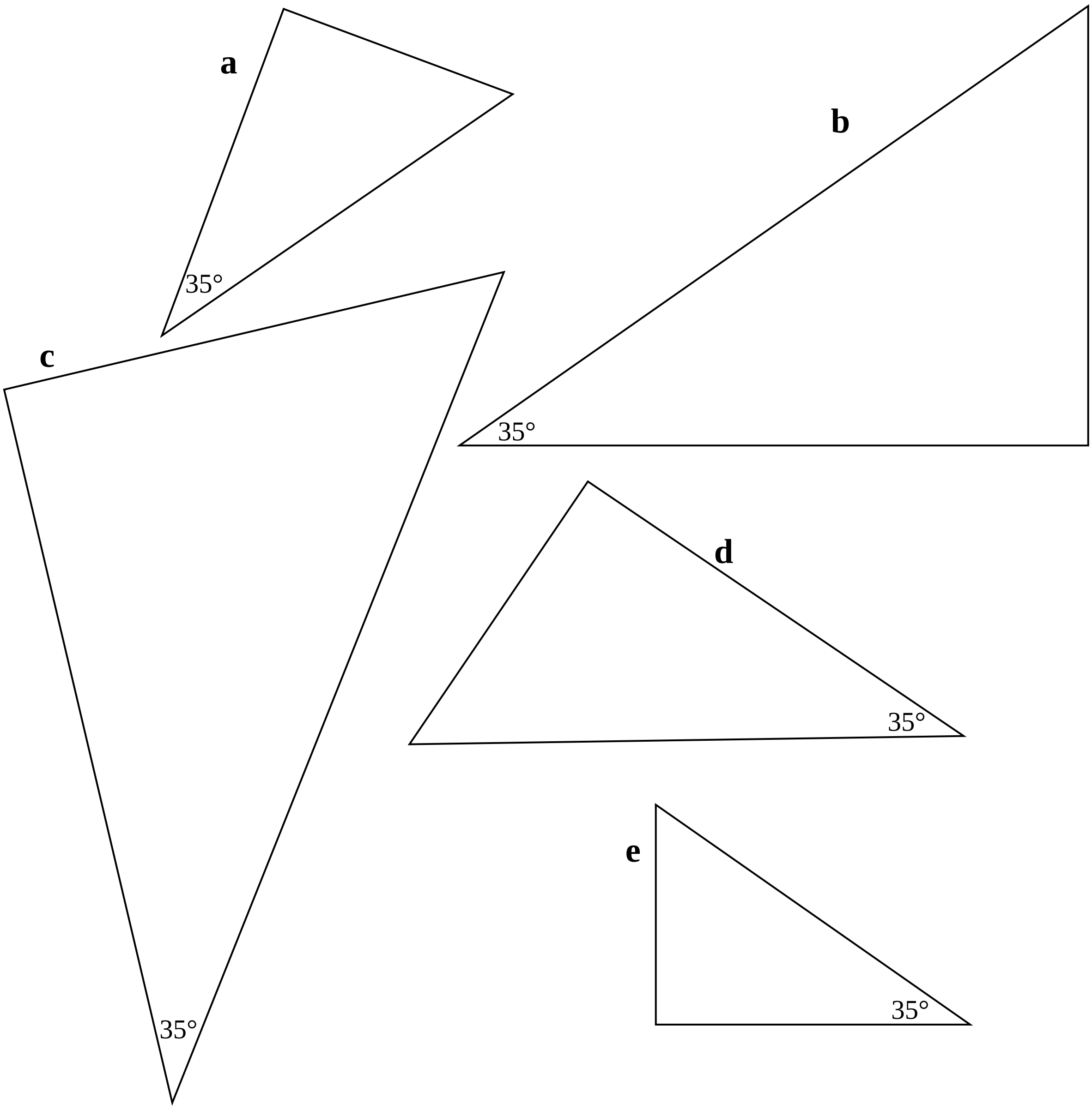
This is quite a steep slope to walk up: you would be unlikely to want to go as steep as 20°. It may not be convenient for wheelchair users.

Trigonometry Investigation

Label the sides of these right-angled triangles with

hyp – hypotenuse (the side opposite the right-angle)	**opp** – opposite (the side opposite the 35° angle)	**adj** – adjacent (the side next to the 35° angle)

Then measure the **opp** and **adj** sides with a ruler (to the nearest mm) and complete the table below.
What do you notice?



triangle	**length of side *opposite* to the 35° angle** (mm, to nearest integer)	**length of side *adjacent* to the 35° angle** (mm, to nearest integer)	**opposite side / adjacent side** (to 2 decimal places)
a			
b			
c			
d			
e			

Trigonometry Facts

LEARN EVERYTHING ON THIS PAGE. LEARN EVERYTHING ON THIS PAGE. LEARN EVERYTHING ON THIS PAGE.

Definitions

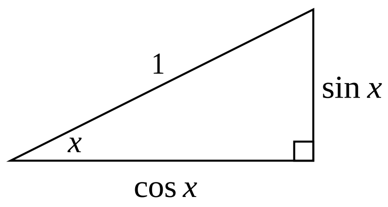
SOHCAHTOA gives the meaning of $\sin x$, $\cos x$ and $\tan x$.

Just remember the other three as

$$\cot x = \frac{1}{\tan x},\ \sec x = \frac{1}{\cos x} \text{ and } \operatorname{cosec} x = \frac{1}{\sin x}.$$

(the ***s*** and the ***c*** go together – ***s***ec with ***c***os and ***c***osec with ***s***in)

Imagine a right-angled triangle with a hypotenuse of 1 and an angle x.
Using SOHCAHTOA, the shorter sides are $\cos x$ and $\sin x$.



So $\tan x \equiv \dfrac{\sin x}{\cos x}$,

and by Pythagoras,

$\sin^2 x + \cos^2 x \equiv 1$.

Dividing by $\sin^2 x$ gives $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$, and
dividing by $\cos^2 x$ gives $\tan^2 x + 1 \equiv \sec^2 x$.

Radians

Always use them for angles unless the context specifically uses degrees.
Calculus will work only in radians.

$$360° = 2\pi,\ 180° = \pi,\ 90° = \tfrac{\pi}{2},\ 45° = \tfrac{\pi}{4}, \text{ etc.}$$

(the ***3*** and the ***6*** go together in $30° = \frac{\pi}{6}$ and $60° = \frac{\pi}{3}$)

Make sure your calculator is in the right mode!

Exact Values

The properties of equilateral and right-angled isosceles triangles give these exact values of $\sin x$, $\cos x$, $\tan x$.
Use them whenever possible so that your answers are exact.

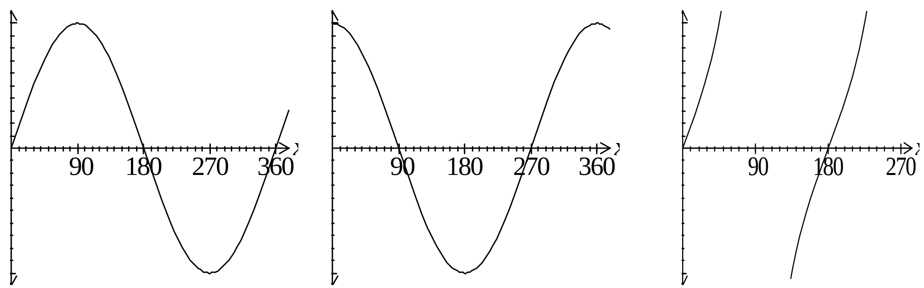
	$0° = 0$	$30° = \frac{\pi}{6}$	$45° = \frac{\pi}{4}$	$60° = \frac{\pi}{3}$	$90° = \frac{\pi}{2}$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$

Notice that $\sin x = \cos(90 - x)$ and $\cos x = \sin(90 - x)$.

Graphs

When you're solving trig equations, finding solutions within a certain range of angles, always draw one of these graphs.

$\sin x$ $\cos x$ $\tan x$



$\sin x$ and $\cos x$ vary between 1 and –1 with a period of 360°.
$\tan x$ can take any value and has a period of 180°.

Use the symmetry of these graphs to find multiple solutions.

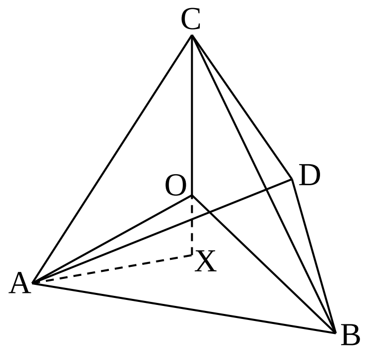
Trigonometry Formulas

Compound Angle Formulas	These are the most important ones – you must learn them.
$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$ $\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$ $\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$ $\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$ $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$	"sin,cos *plus* cos,sin" "cos,cos *minus* sin,sin" Remember the *minus* in the $(A+B)$ formulas for $\cos$ and $\tan$.
Double Angle Formulas	These come from letting $B = A$ in the above.
$\sin 2A \equiv 2\sin A\cos A$ $\cos 2A \equiv \cos^2 A - \sin^2 A$ $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$	These are also worth learning separately because they're so useful.
Formulas Using Double Angles	These come from combining the double angle formulas with the identity $\sin^2 A + \cos^2 A \equiv 1$.
$\cos 2A \equiv 2\cos^2 A - 1$ $\cos^2 A \equiv \frac{1}{2}(1 + \cos 2A)$ $\cos 2A \equiv 1 - 2\sin^2 A$ $\sin^2 A \equiv \frac{1}{2}(1 - \cos 2A)$	You could work these out (or look them up) when you need them, but you need to know that they exist.
Factor Formulas	
$\sin A + \sin B \equiv 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$ $\sin A - \sin B \equiv 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$ $\cos A + \cos B \equiv 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$ $\cos A - \cos B \equiv -2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$	These are very useful, and you certainly don't want to have to work them out. Either learn them or rely on looking them up when you need them. Remember the *"minus* $\sin s$*"* in the last one.
Formulas for Triangles	a, b and c are the lengths of the sides; A, B and C the angles opposite the 3 sides.
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2r$	This is the *Sine Rule*. r is the radius of the circumscribed circle.
$a^2 = b^2 + c^2 - 2bc\cos A$	This is the *Cosine Rule*.
area $= \frac{1}{2}ab\sin C = \sqrt{s(s-a)(s-b)(s-c)}$	s is the semi-perimeter $= \frac{1}{2}(a+b+c)$.

Calculating the H-C-H bond angle in Methane (CH_4)

(Uses Pythagoras' Theorem)

Imagine a tetrahedron ABCD with unit edge length and a hydrogen atom at each of the vertices. (In the diagram below, ABD is the base.)
The position O is the location of the carbon atom, and is equidistant from each of the vertices.
We wish to find angle AOB.



First we can find AX.
Looking from above, X is just the centre of the bottom equilateral triangle, so using trigonometry in this triangle we find that AX = $\frac{1}{2} \div \cos 30 = \frac{1}{\sqrt{3}}$.

Now the total height of the tetrahedron, CX, is a shorter side of the right-angled triangle ACX, so using Pythagoras' Theorem we find that CX = $\sqrt{AC^2 - AX^2} = \sqrt{1 - \left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{2}{3}}$.

We next find the C-H bond length l, which is equal to OA or OC.
Using Pythagoras' Theorem in triangle OAX we have

$$\begin{aligned} OA^2 &= AX^2 + OX^2 \\ &= AX^2 + (CX - OC)^2 \\ &= \tfrac{1}{3} + CX^2 - 2 \times CX \times OC + OC^2 \\ l^2 &= \tfrac{1}{3} + \tfrac{2}{3} - 2\sqrt{\tfrac{2}{3}}\,l + l^2 \end{aligned}$$

So $2\sqrt{\frac{2}{3}}\,l = 1$ and so $l = \sqrt{\frac{3}{8}}$.
(Subtracting l^2 from both sides is OK here.)

Finally we use the cosine rule in triangle OAB to find angle AOB.

$$\cos AOB = \frac{OA^2 + OB^2 - AB^2}{2 \times OA \times OB} = \frac{2l^2 - 1^2}{2l^2} = \frac{2\left(\frac{3}{8}\right) - 1}{2\left(\frac{3}{8}\right)} = \frac{-\frac{1}{4}}{\frac{3}{4}} = -\tfrac{1}{3}$$

So angle AOB = $\cos^{-1}(-\frac{1}{3}) = 109.47$ °.

There are other ways of arriving at the value of this angle.