2.7 Pythagoras' Theorem

- You could say that this is really "Trigonometry" because it's to do with solving triangles.
- It's a good opportunity to revise circles, because there are so many good problems applying Pythagoras' theorem to circles, arcs, spheres, etc. (see later).
- Hypotenuse is the side opposite the right-angle. This may be a better definition than "the longest side", because this way it's clear that there isn't one in a non-right-angled triangle – any scalene triangle and some isosceles triangles will have a "longest" side.)
- Instead of writing $a^2+b^2=c^2$ or $a^2=b^2+c^2$ and having to remember which letter is the hypotenuse, pupils could write it as $\text{hyp}^2=a^2+b^2$.
- The *converse* of Pythagoras is sometimes omitted, but provides a good opportunity to discuss the concept of "converse" and to think of examples of when the converse of something is true and when it isn't. If A and B are the following statements, A = "the triangle is right-angled", B = "the square of the hypotenuse is equal to the sum of the squares on the other two sides", then Pythagoras' theorem is the conclusion that A implies B ($A \Rightarrow B$). The converse is that $B \Rightarrow A$, that any triangle in which statement B is true must be right-angled. So in this case $A \Leftrightarrow B$ (A is equivalent to B), but in general if $A \Rightarrow B$, B doesn't necessarily imply A. One example is if A = "the triangle is right-angled" and B = "the shape has exactly three sides". Here $A \Rightarrow B$ but $B \not\Rightarrow A$ because although all right-angled triangles have three sides, not all triangles are right-angled.
- In three dimensions $a^2=b^2+c^2+d^2$. This makes sense by seeing that b^2+c^2 is the square of the hypotenuse of the right-angled triangle in the plane defined by sides b and c (the plane perpendicular to side d). Then applying 2-d Pythagoras' theorem again gives the result.
(You could just as well start with c and d or with b and d.)
- Pythagoras' Theorem is so powerful because it is readily applied to more complicated circumstances than a single right-angled triangle; e.g., any non-right-angled isosceles triangle can be cut into two congruent right-angled triangles.
- In solving right-angled triangles, it's helpful to distinguish between finding the *hypotenuse* (square, add, square root) and finding one of the shorter sides (sometimes called *legs*) (square, subtract, square root).

2.7.1 **NEED** 1 cm × 1 cm squared dotty paper (see section 2.1). Tilted Squares.
We are going to draw tilted squares on square dotty paper so that each vertex lies on a dot.
Start by drawing a "$\begin{pmatrix}2\\1\end{pmatrix}$ square", where the lowest side goes 2 along and 1 up (gradient of $\frac{1}{2}$). Work out the area.

Answer:

If pupils get stuck drawing the squares, they can say "2 along-1 up, 1 along-2 up, 2 along-1 down, 1 along, 2 down" as they go around the square.

area = 5 cm².
There are various ways of cutting up the square. (Some pupils may prefer to measure the sides with a ruler as accurately as they can and find the area that way, although it may be less "elegant".)

Try other $\begin{pmatrix}x\\1\end{pmatrix}$ squares and look for a pattern.

area = x^2+1

Then try $\begin{pmatrix}x\\y\end{pmatrix}$ squares.

area = x^2+y^2

Then focus in on the right-angled triangle "underneath" the square. What are these results telling us about the sides of the triangle?

If the area of a square is 36 cm², then how long are the sides? etc.

2.7.2 See if Pythagoras' Theorem works for all triangles (see "Triangles and Tilted Squares" sheet).
Answers:

A	obtuse-angled (25 > 9 + 10)

The square on the longest side equals the sum of the squares on the other two sides only if the triangle is right-angled.
If it is obtuse-angled, then the square on the longest

B	right-angled (26 = 8 +18)
C	acute-angled (16 < 13 + 13)
D	right-angled (25 = 5 + 20)
E	obtuse-angled (9 > 2 + 5)

Try other shapes on the sides of right-angled triangles; e.g., semicircles or equilateral triangles.

side (the one opposite the obtuse angle) is greater than the sum of the squares on the two other sides. If it is an acute-angled triangle, then the square of any of the sides is smaller than the sum of the squares on the other two.

Any shape will work provided that the shapes on each side are mathematically similar. This corresponds to multiplying the equation $\text{hyp}^2 = a^2 + b^2$ by a constant.

2.7.3 How many different sized squares can you draw on a 3 × 3 dotty grid if every vertex has to lie on a dot? What if you use a larger square grid?

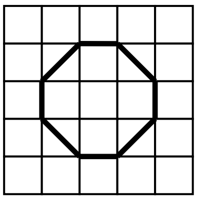
(See the similar investigation using triangles and quadrilaterals in "Polygons" section.)

The pattern is broken at n = 8 because the tilted squares with sides of gradient $\pm\frac{3}{4}$ *and* $\mp\frac{4}{3}$ *have sides with length 5 units, matching one of the untilted squares already counted. This will happen whenever you get Pythagorean Triples.*

Answer: 3
For an $n \times n$ *grid of dots,*

n	***no. of different sized squares***
1	*0*
2	*1*
3	*3*
4	*5*
5	*8*
6	*11*
7	*15*
8	*18 (yes)*

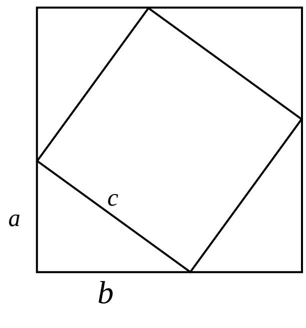
2.7.4 Why is this shape *not* a regular octagon?



Answer:
Although all the interior angles are all 135°, there are four sides of unit length and four "diagonal" sides of longer length $\sqrt{2}$ *units.*

This question aims to confront a common misconception.

2.7.5 Proof: there are a vast number of them.
The simplest is probably the one equating areas in the diagram below (see right).



There are a large number of different proofs of Pythagoras' Theorem: pupils could search on the internet for other proofs.

This is a square with sides of length c inside a square with sides of length $a+b$.
The area of the large square can be worked out in two different ways, and they must give the same answer.

First method: area = $(a+b)^2 = a^2 + 2ab + b^2$.
(If pupils are not familiar with this result from algebra, you can divide up the large square into two congruent rectangles, each of area ab, and two squares of different sizes, a^2 and b^2.)

Second method: add up the areas of the smaller square and the four congruent right-angled triangles; so
area = $c^2 + 4 \times \frac{1}{2}ab = c^2 + 2ab$.
So $a^2 + b^2 + 2ab = c^2 + 2ab$, and therefore $a^2 + b^2 = c^2$ (Pythagoras' Theorem).

2.7.6 Pythagorean Triples.
The lengths of the sides of integer right-angled triangles.
(3,4,5); (5,12,13) and (7,24,25) are the simplest.
Any numbers ($2pq$, $p^2 - q^2$, $p^2 + q^2$) will always

Some pupils could write a BASIC (or similar) computer program to find Pythagorean Triples.

A "primitive" Pythagorean Triple is one that isn't just a scaling up of a similar smaller triangle by multiplying all the sides by the same amount; i.e., it's work (where p and q are integers and $p > q$).

one in which the sides are pairwise co-prime. To get only these, p and q must be co-prime and of opposite "parity" (one odd, the other even).

You can prove that these sides satisfy Pythagoras' formula by squaring and adding:

$$(p^2 - q^2)^2 + (2pq)^2$$
$$= p^4 - 2p^2q^2 + q^4 + 4p^2q^2$$
$$= (p^2 + q^2)^2$$

A method equivalent to this for generating Pythagorean triples is to add the reciprocals of any 2 consecutive odd or consecutive even numbers. The numerator and denominator of the answer (whether simplified or not) are the two shorter sides of a right-angled triangle; e.g.,

$\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$, giving the (7,24,25) triangle.

This works because $\frac{1}{p-1} + \frac{1}{p+1} = \frac{p+1+p-1}{p^2-1} = \frac{2p}{p^2-1}$, as above where $q = 1$.

2.7.7 Pythagorean Triples.

1. Investigate the factors of the numbers in "primitive" Pythagorean triples.
2. Investigate the product of *the two legs* of a Pythagorean triple.
   Try for different Pythagorean triples.
3. Investigate the product of *all three sides* of a Pythagorean triple.
   Try for different Pythagorean triples.

Answers:

1. *In "primitive" triples (see above), the largest number is always odd, and of the other two one is odd and one is even. One of the sides is always divisible by 2, one by 3 and one by 5 (see below). (These may all be the same side; e.g., 60 in 11, 60, 61.)*
2. *The product of the two legs is always a multiple of 12 (or, equivalently, the area is always a multiple of 6). (Interestingly, this area can never be a square number.)*
   You can prove this using the expressions above: area = $\frac{1}{2} \times 2pq \times (p^2 - q^2) = pq(p^2 - q^2)$.
   Here, either p is even or q must be, so pq is a multiple of 2. If p or q are multiples of 3, then the whole thing will be a multiple of 6. If neither p nor q is a multiple of 3, then $p = 3r \pm 1$ and $q = 3s \pm 1$, so $p^2 - q^2$ must be a multiple of 3 (multiply out p^2 and q^2 and the "+1's" cancel out), so the whole area is still a multiple of 6. Because the product of the legs is $2pq(p^2 - q^2)$, then this number will be a multiple of 12.

Answers (continued):

3. *The product of all three sides is always a multiple of 60. To show this, in addition to the above argument we need to show that $pq(p^2 - q^2)(p^2 + q^2)$ is a multiple of 5.*
   If either p or q is a multiple of 5, then clearly the whole thing will be. If neither is, then $p = 5r \pm 1$ or $p = 5r \pm 2$ and $q = 5s \pm 1$ or $q = 5s \pm 2$. When you multiply out p^2 and q^2, you find that they are either 1 more or 1 less or 4 more or 4 less than multiples of 5. So it is always the case that either $p^2 + q^2$ or $p^2 - q^2$ is a multiple of 5, so the whole thing must be.

2.7.8 Fermat's Last Theorem.

We have some solutions to $a^2 + b^2 = c^2$ where a, b and c are positive integers (see Pythaogrean Triples above).
Try to find solutions to these equations:

$$a^3 + b^3 = c^3$$
$$a^4 + b^4 = c^4$$

Pupils could look for solutions to $a^2 + b^2 + c^2 = d^2$.

Fermat (1601-1665) believed that there were no solutions to the equation $a^n + b^n = c^n$ where $n > 2$ and a, b and c are positive integers. He claimed to have a proof but never wrote it down. It has since been proved using highly complicated maths.

There are many "Pythagorean Quadruples"; e.g., (1, 2, 2, 3); (1, 4, 8, 9); (9, 8, 12, 17).
This time, any numbers ($2pr$, $2qr$, $p^2 + q^2 - r^2$, $p^2 + q^2 + r^2$) work, where $p, q, r > 0$.

2.7.9 **NEED** keyboard diagrams (see sheets).
Keyboard Typing.
Imagine typing words with 1 finger. Say that each key is 1 cm × 1 cm. How far does your finger have to move to type certain words?
(Calculate from the centre of each key.)
e.g., a word like FRED is easy.
What about HELP?
What four-letter word has the longest distance on the keyboard?

A different way of thinking of "word length".
(The keys on the diagram are 1.5 cm × 1.5 cm to discourage measuring.)

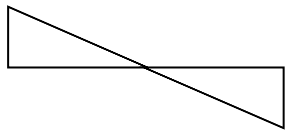
Answers:
FRED = 3 cm (F-R, R-E and E-D)

HELP = 3.64 + 6.58 + 1.12 = 11.34 cm
(using Pythagoras' Theorem)

ZONE = 14.58 cm, but there may be longer words.

2.7.10 Two football players start side by side. They each run 4 m in a straight line, turn 90° to the right and run another 3 m, again in a straight line. What is the furthest apart they could now be?

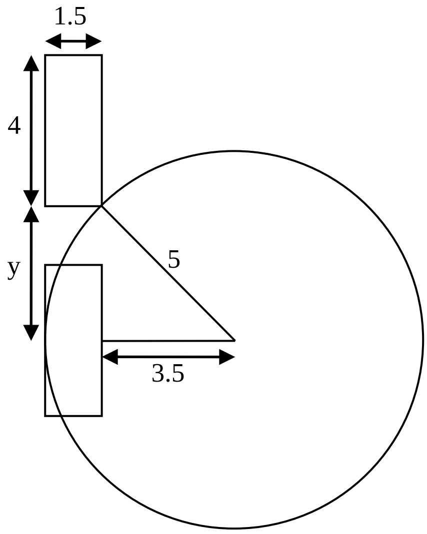
Answer: 10 m, if they started facing in opposite directions. (Imagine two 3-4-5 right-angled triangles meeting at the players' starting point.)



2.7.11 A straight road contains a row of parked cars all of width 1.5 m and length 4 m. If one of the cars has a turning circle of 10 m, how much space will it need in front of it so that it can pull out without having to reverse?

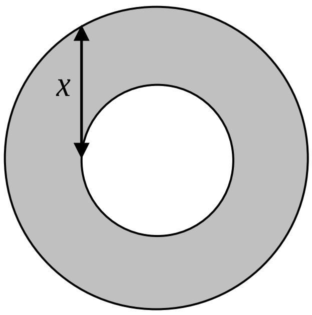
What assumptions do you have to make?
(A turning circle of 10 m kerb-to-kerb means that the car can just manage a U-turn at slow speed in a street 10 m wide.)

Answer:
In the diagram to the right, the distance from the centre of the turning circle to the offside of the car is 5 – 1.5 = 3.5 m. Applying Pythagoras' Theorem, $y^2 = 5^2 - 3.5^2$, *giving* y *= 3.6 m, measured from the mid-point of the length of the car. The necessary distance in front of the car is therefore 3.6 – 2 = 1.6 m.*



We have assumed that the cars are parked exactly in line, that both cars have the same width and that the driver gets full right-lock as soon as the car begins to move.

2.7.12 In the diagram below, find the shaded area in terms of x.



Answer:
If the large circle has radius R *and the small circle,* r, *then the shaded area =* $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$.
But by Pythagoras' Theorem, $x^2 = R^2 - r^2$, *so the shaded area =* πx^2.

2.7.13 A ladder of length 13 feet is standing upright against a wall. If the top end slides down the wall 1 foot, how far out from the wall will the bottom end move?

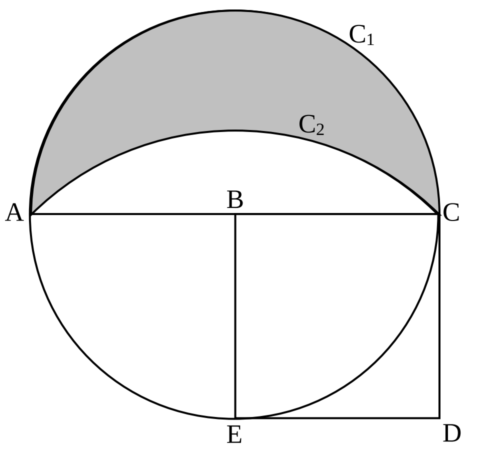
Answer:
5 feet (a 5-12-13 right-angled triangle)

2.7.14 A cable 1 km long is lying flat along the ground with its ends fixed. If its length is increased by 1 m but the ends are still fixed 1 km apart, how high up can the mid-point of the cable be raised before it becomes taut? (Assume the cable doesn't stretch or sag.)

Answer: 22.4 m (a surprisingly large amount). The shape produced is a (very) obtuse-angled isosceles triangle. Each half is a right-angled triangle with hypotenuse 500.5 m and base 500 m. Calculating the third (vertical) side gives the answer.

2.7.15 In the diagram below, C_1 is a semicircular arc centred on B and C_2 is a quarter-circular arc centred on E. Prove that the area of the shaded *lune* between C_1 and C_2 is equal to the area of the square BCDE.

Answer:
Let BC = r.
Then the area of the square BCDE = r^2.
Area of semicircle C_1 = $\frac{1}{2}\pi r^2$, *and area of quadrant ACE =* $\frac{1}{4}\pi(\sqrt{2}r)^2 = \frac{1}{2}\pi r^2$.



Area of triangle ACE = $\frac{1}{2}(\sqrt{2}r)^2 = r^2$ *(since angle AEC = 90°, angle in a semicircle), so area of segment =* $\frac{1}{2}\pi r^2 - r^2$.
Therefore area of shaded lume = $\frac{1}{2}\pi r^2 - (\frac{1}{2}\pi r^2 - r^2) = r^2$ *= area of square BCDE, as required.*

This and other similar results were discovered by Hippocrates of Chios (470-410 BC).

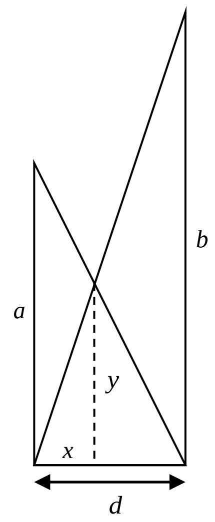
2.7.16 A rope is attached to the top of a vertical pole and at the bottom 1 m is lying on the ground. When the end of the rope is pulled along the ground until it is taut, its end is 5 m from the base of the pole. How long is the rope and how high is the pole? (The rope doesn't stretch.)

Answer: The rope is 13 m long and the pole is 12 m high (5, 12, 13 triangle).
If h *is the height of the pole, then* $(h+1)^2 = h^2 + 5^2$, *and solving this equation gives these values.*

2.7.17 A narrow passageway of width 1 m contains two ladders leaning against the walls. Each has its foot at the bottom of one wall and its top at the top of the other wall. If the walls have heights 2 m and 3m, how high above the ground is the point where the ladders cross?

Answer: 1.2 m
One approach is to model the ladders as segments of the lines $y = 3x$ *and* $y = -2x + 2$. *Solving simultaneously gives* $x = 0.4, y = 1.2$.

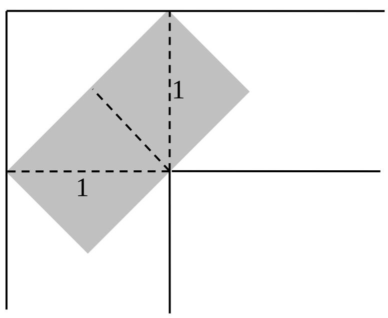
Alternatively, you can use similar triangles. Using the letters as defined on the right,
$\frac{y}{x} = \frac{b}{d}$ *and* $\frac{y}{d-x} = \frac{a}{d}$, *so eliminating y,*
$\frac{bx}{d} = \frac{a(d-x)}{d}$ *so that* $x = \frac{ad}{a+b}$ *and* $y = \frac{ab}{a+b}$, *so the answer to the original question didn't depend on* d *(1 m).*



Similar-looking problems where the lengths of the ladders are given instead of vertical heights are much harder, often leading to quartic equations.

2.7.18 A piano in a cuboid crate has to be moved round a right-angled corner in a corridor of width 2 m. If no part of the crate is lifted off the floor, what are the dimensions of the biggest crate that will just go round the corner? Assume that the floor is perfectly horizontal and the walls perfectly vertical.

Answer: The tightest squeeze will happen when the crate is positioned as below.

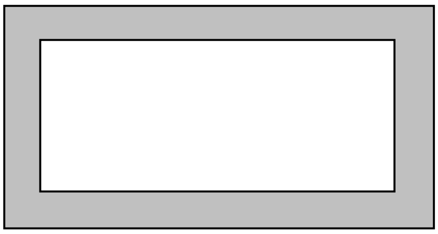


If *the crate just fits, then its sides will be* $\frac{\sqrt{2}}{2}$ *by* $\sqrt{2}$ *, so it's area will be 1 m²*.

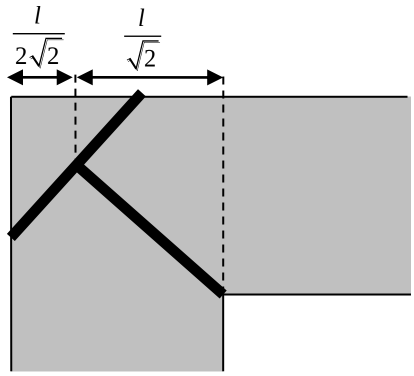
2.7.19 A border of 1 m width around a rectangular garden is

Answer:

covered with wet cement. You have two 98 cm long narrow planks of wood. Can you use them to bridge across the cement from the outside to the inside?



Just, if you put them across at the corner. If l is the length of the planks, and they just meet in the arrangement shown below, then the distances marked are as shown.



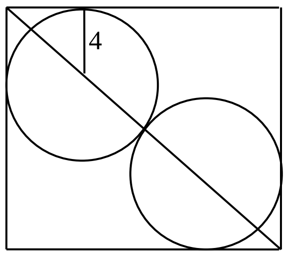
Therefore, $\frac{l}{2\sqrt{2}}+\frac{l}{\sqrt{2}}=1$, *giving* $\frac{3l}{2\sqrt{2}}=1$, *so*

$l=\frac{2\sqrt{2}}{3}=0.94$ *m (2 dp).*

But you would need to allow a bit for overlapping of the planks and the grass.

2.7.20 Four identical apples of diameter 8 cm have to be fitted into a cubical box. What is the smallest box that will do? Assume that the apples are perfect spheres.

Answer:
The apples need to be stacked "tetrahedrally" so that there are 2 along the diagonal of the bottom of the box and 2 along the other diagonal in the top half of the box.



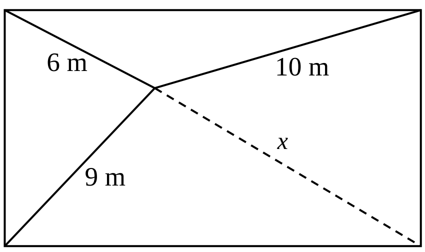
You can see that the diagonal of the box has length $4\sqrt{2}+4+4+4\sqrt{2}$*, so the sides of the box have length* $\frac{8}{\sqrt{2}}(1+\sqrt{2})=4\sqrt{2}+8$ *= 13.7 cm.*

2.7.21 Which fits better, a square peg in a round hole or a round peg in a square hole?

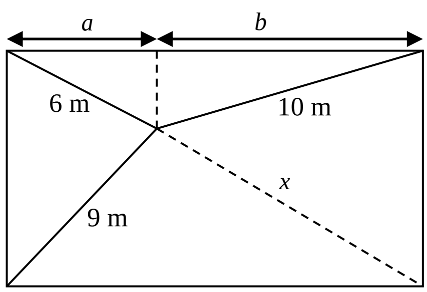
Answer:
A round peg in square hole occupies

$\frac{\pi r^2}{(2r)^2}=\frac{\pi}{4}=79\%$ *of the square, and this is better than a square peg in round hole, which occupies only* $\frac{\left(\frac{2r}{\sqrt{2}}\right)^2}{\pi r^2}=\frac{2}{\pi}=64\%$ *of the circle.*

2.7.22 I am standing in a rectangular hall and my distances from three of the corners are as shown below. How far must I be from the fourth corner?



Answer:
Let a *and* b *be as shown below.*



Then using Pythagoras' theorem in the top two right-angled triangles, we can equate expressions for the vertical dashed line, giving $6^2-a^2=10^2-b^2$.

Similarly for the bottom part of the diagram, $9^2 - a^2 = x^2 - b^2$, and subtracting these equations leaves $9^2 - 6^2 = x^2 - 10^2$, and solving this gives $x = \sqrt{145} = 12.04$ m.

2.7.23 How big is the smallest circle which you can fit a 2 cm by 4 cm rectangle into?

Answer:
The widest length in the rectangle will be the diagonal, which is $\sqrt{2^2 + 4^2} = \sqrt{20}$ cm, so that will have to be the diameter. So the radius of the circle will be $\frac{1}{2}\sqrt{20} = \sqrt{5} = 2.24$ cm.

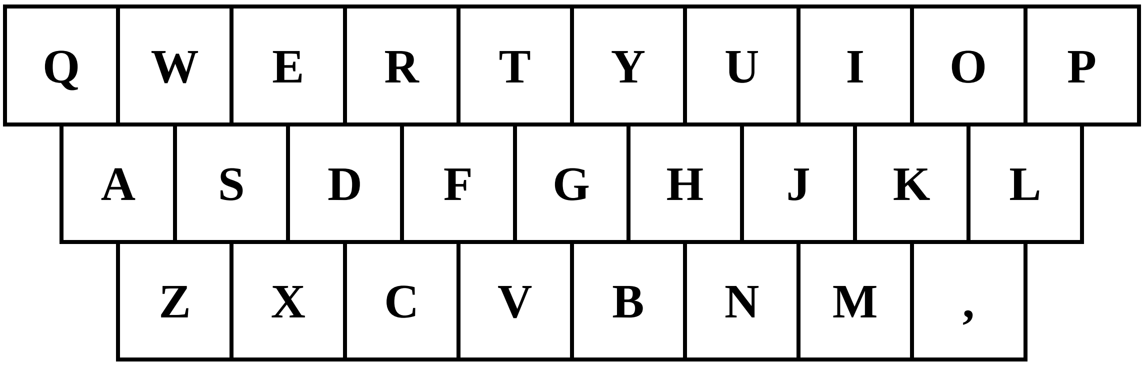
Triangles and Tilted Squares

Plot these triangles on axes going from 0 to 20 both horizontally and vertically.
(With a bit of overlapping, they will all fit onto one set of axes.)

Label them A to E, and note what *kind* of triangle each one is.

Draw tilted squares on each side and work out their areas.
Look at your results for each triangle.
What do you notice?

A	(10, 7)	(9, 4)	(6, 4)
B	(9, 15)	(4, 14)	(7, 17)
C	(14, 7)	(17, 5)	(14, 3)
D	(14, 16)	(18, 14)	(17, 12)
E	(2, 3)	(3, 5)	(3, 2)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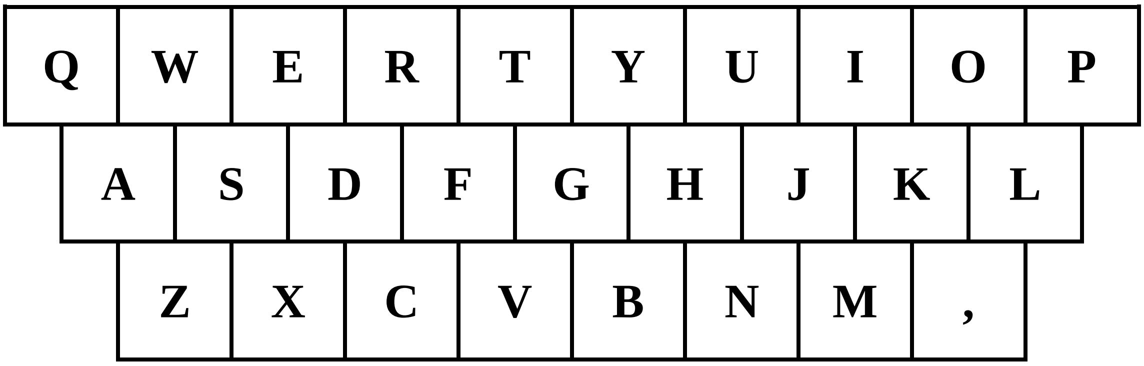
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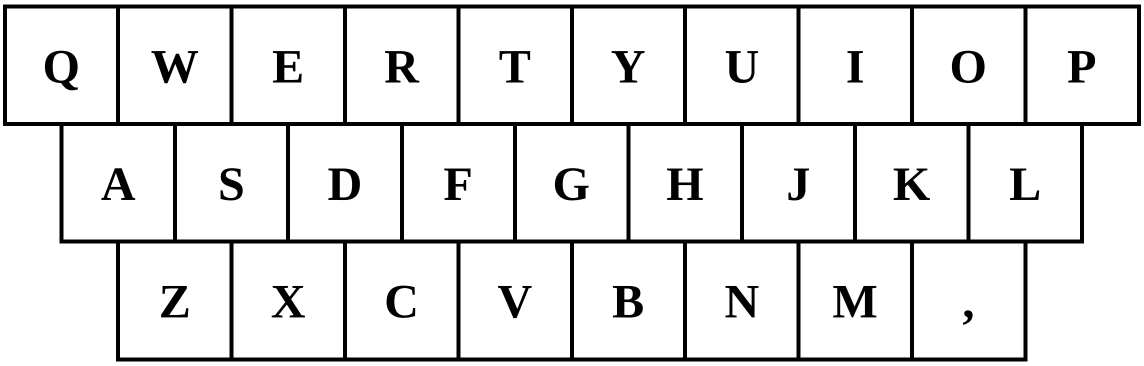
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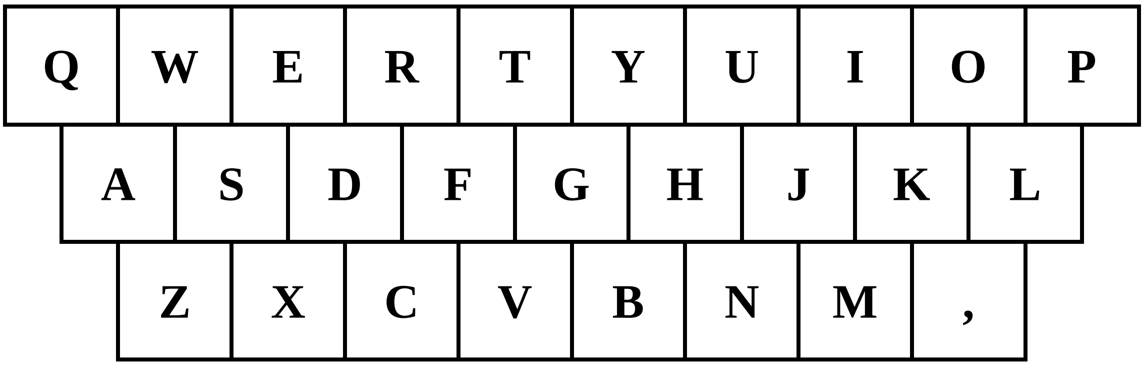
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