2.10 Volume

- If you're using the words "solid" for 3-d and "shape" for 2-d, pupils need to realise that, of course, you can work out volumes of liquids and gases as well as solids. (In particular, volumes of solids and liquids are more or less constant whereas gas volume depends on pressure, temperature, etc. and not just on the mass.) *Capacity* is just the volume of space (or air) inside a "hollow solid".
- There is some overlap with section 2.15, but the following is pretty essential:

			÷1000		÷1000	
			→		→	
cm^3	=	**ml**		**litre**		**m^3**
			←		←	
			×1000		×1000	

- Some opportunity to handle and count cubes is essential in the early stages of this topic. Cubes which can be fitted together to make larger cubes/cuboids/etc. are ideal.

2.10.1 Words that mean different things in maths from what they mean in ordinary life or other subjects.
Think of some examples.

In maths/science, volume means how much space something takes up or how much space there is inside something (sometimes called *capacity*).

Take-away, difference, product, factor, prime, negative, positive, sign, odd, even, root, index, power, improper, rounded, interest, expression, identity, solution, term, subject, acute, obtuse, reflex, face, net, square, plane, prism, compasses, translation, sketch, origin, arc, chord, similar, tangent, mean, range, raw, frequency, certain, impossible, independent, etc.
Also volume (loudness in science, vague "amount" in common usage, or can refer to a book).

2.10.2 **NEED** cubes, common cuboid objects.
How many cubes make up this cube/cuboid?
You can show $2 \times 2 \times 2$ and $2 \times 3 \times 4$ etc. cuboids to see that volume means the number of cm^3 that will fit inside. Hence multiply the three dimensions to find the volume.

If the cubes you're using aren't cm^3 you can say that you're imagining they are.
Stick with integer lengths at this stage.

This is really
volume = area of one layer × number of layers.

Find the volume, by measuring the dimensions, of common objects: maths book, video cassette, briefcase, locker, room?
Start by estimating how many cm^3 would go into it.

typical values: (pupils tend to underestimate)
exercise book: 100 cm^3; textbook: 1000 cm^3 (we're learning a litre of maths this year!)
video cassette: 400 cm^3; briefcase: 30 litres; locker: 70 litres (roughly the volume of a human being – some pupils will fit inside their lockers, but don't try it!).

2.10.3 If we woke up tomorrow and everything had doubled in size, would there be any way to tell? (Poincaré, 1854-1912, originally posed this famous riddle.)

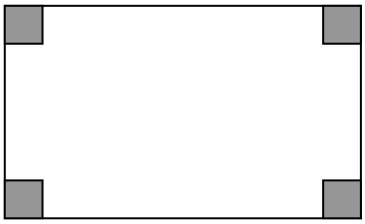
More precisely we mean if every length doubled (so 5 cm became 10 cm and so on), because of course that would mean that area had become four times as much and volume eight times as much.

It depends whether other things besides length changed as well. Presumably things would look *the same because our own eyes and bodies would be twice as large (so perspective effects would be the same), but if there were no corresponding increase in mass (for example) it would be easy to detect, because, for example, gravity would be weaker. To make it work, sub-atomic forces would have to increase too.*

2.10.4 **NEED** A4 1 cm × 1 cm squared paper, scissors, sticky tape. Maximum Volume from a piece of paper.

This size makes for easier calculations than using

Cut out an 18 cm by 24 cm rectangle from a sheet of A4 paper.
We want to make an open box (no lid) out of this paper that has the maximum possible volume. We'll cut out squares from each corner and see what is the maximum volume we can get.
(Imagine you were collecting sweets in it from a big boxful at the front of the room!)



(diagram not to scale)

Start by cutting off 1 cm × 1 cm squares from each of the four corners. Fold up the sides to make a very shallow box.

Making a cone out of the paper would probably give a smaller volume. (If you cut out a quadrant of radius 18 cm you could roll this up into a cone of slant height 18 cm and base radius $18 \div 4$ *= 4.5 cm, so the volume would be about 350 cm*^{3}*.)*

actual A4 size.

Let x *cm be the length of the side of the square cut off. Then we get the following results:*

x	*dimensions of box*	*volume (cm*^{3}*)*
1	*16 × 22 × 1*	*352*
2	*14 × 20 × 2*	*560*
3	*12 × 18 × 3*	*648*
4	*10 × 16 × 4*	*640*
3.4	*11.2 × 17.2 × 3.4*	*655*

Could plot a graph of volume against x *, but you can see from the numbers that the maximum is between 2 and 4. Trial and improvement gives* $x = 3.4$ *cm (1 dp).*

It's possible to get the same answer using calculus:
$V = x(24-2x)(18-2x) = 4x^3 - 84x^2 + 432x$ *, so differentiating,*
$\frac{dV}{dx} = 12x^2 - 168x + 432 = 0$ *for stationary points, and solving this quadratic gives* $x = 3.4$ *as the only solution in the range* $0 < x < 9$ *.*

2.10.5 **NEED** interlocking cubes.
Minimum Surface Area for a given Volume (the above problem in reverse).
This could be introduced as the problem of wrapping up a number of identical cubes so as to use the minimum amount of wrapping paper. "What's the best shape for a packet of sugar lumps?" would be a more open-ended problem.

Think of a situation where maximum or minimum surface area is important.

Keeping warm (huddle up – minimise surface area); getting a sun-tan (spread out – maximise surface area).
Surface tension causes soap bubbles to minimise their surface area (pupils may have seen a demonstration in Science).
Lungs have a very large surface area (over 100 m^2*) because that's where oxygen is absorbed.*
Granulated sugar dissolves faster than sugar lumps because the water molecules have more exposed sugar to bump into.

The surface area of a solid is the area of its net (excluding any "tabs"), if it has one. (A sphere has a surface area although it has no net.)

"Best" would mean not just the minimum amount of cardboard; you'd have to consider how the packet would look, how easy it would be to fit the design and details on the packet, how stable it would be, etc.

The minimum surface area is obtained when the cubes make a solid that is nearest to a cube in shape (see below).

no. of cubes	***max surface area***	***min surface area***
24	*1 × 1 × 24:* ***98***	*2 × 3 × 4:* ***52***
27	*1 × 1 × 27:* ***110***	*3 × 3 × 3:* ***54***
48	*1 × 1 × 48:* ***194***	*3 × 4 × 4:* ***80***
64	*1 × 1 × 64:* ***258***	*4 × 4 × 4:* ***96***

Maximum surface area comes from arranging the cubes in a long line (a prism with cross-section 1 × 1). In fact, it doesn't affect the surface area if the "line" has bends in it, but then the solid isn't a simple cuboid any more.

In general, for n *cubes, the maximum surface area =* $4n + 2$.

2.10.6 Length comparisons versus volume comparisons. People often choose length comparisons to make something seem a lot and volume comparisons to make something not seem that much.

How many 10 p coins would you need to make a pile all the way to the top of Mount Everest (8800 m)? Assume each coin is 1 mm thick.

There are issues here of misleading statistics.

Answer:

How big a container would you need to put them in?

$\frac{8800}{10^{-3}} = 8.8\times10^6$ coins (= £ 880 000).

Each 10 p coin would fit inside a cuboid box 1 mm × 25 mm × 25 mm, which is a volume of $(1\times10^{-3})\times(25\times10^{-3})\times(25\times10^{-3}) = 6.25\times10^{-7}$ m³.
So all of these coins will take up only $(8.8\times10^6)\times(6.25\times10^{-7}) = 5.5$ m³; i.e. a cube box with sides 1.75 m (not that big).

You can do a similar thing with people.
If you lined up all the people in the world end to end, how far would they stretch?
(Assume that they are lying down end-to-end.)

Assuming that there are about 6.5×10^9 people in the world, and taking an average height of 2 m, the distance would be $(6.5\times10^9)\times2 = 1.3\times10^{10}$ m.
The average distance to the moon is about 4×10^8 m, so this is $\frac{1.3\times10^{10}}{4\times10^8} = 32.5$, so they would stretch to the moon and back 16 times.

What if you put each person in a room 5 m by 5 m by 5 m? How much space would they all take up?

Each room would have a volume of $5\times5\times5 = 125$ m³, so for 6.5×10^9 people we would need $(6.5\times10^9)\times125 = 8.1\times10^{11}$ m³.
This is a cube box with sides of length about $\sqrt[3]{8.1\times10^{11}} = 9.3$ km.
So a box about 10 km × 10 km × 10 km (not that big) could contain rooms for all the people in the world!

2.10.7 If all the ice in the Antarctic were to melt, how much higher would the oceans rise?

Approximate volume of glacier ice in Antarctica = 3×10^8 km³;
Approximate ocean surface area = 5×10^9 km².

Answer:
If it all melted, the rise would be about $\frac{3\times10^8}{5\times10^9} = 0.06$ km = 60 m.

This ignores many important factors, such as the fact that water is slightly more dense than ice and also that some of the ice is underwater. (When floating ice melts, the water level doesn't change.)

2.10.8 Given that helium has a lifting power of about 1 gram per litre, how many fairground-type balloons do you think it would take to lift an average person?

This value comes from the densities of helium and air. 1 litre of helium has a mass of 0.18 g, whereas 1 litre of air has a mass of 1.28 g. So by Archimedes' principle the difference of about 0.01 N (equivalent to 1 g) is the resultant upward force.

Answer:
We can assume that each balloon is approximately a sphere with a diameter of about 30 cm. Therefore the volume of helium = $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi\times15^3 = 14130$ cm³, or about 14 litres. So each balloon will lift about 14 g.
An average person of mass 70 kg would therefore need about $70 \div 0.014 = 5000$ balloons (rather a lot!).

2.10.9 Comparing volume and surface area.
What have all these facts got in common?

1. Babies need blankets to keep warm.
2. A mouse can fall a long way and not be harmed.
3. If an ant were enlarged to the size of an elephant it would collapse under its own weight. (That's why elephants have proportionately wider feet.)

Area scale factor of enlargement = x^2;
Volume scale factor = x^3, and assuming constant density the mass would increase by the same factor.

Answer:
1. *The amount of heat that human beings can store is roughly proportional to their volume, but the rate at which they lose heat to their surroundings is roughly proportional to their surface area. Being small, babies have a large ratio of surface area to volume.*
2. *The amount of energy the mouse has when it hits the ground is proportional to its mass (or its volume) but the area of impact is proportional to its surface area, so having a lot of surface area for its volume (because it's small) helps.*
3. *If the linear scale factor of enlargement was x,*

(Stiletto heels damage some floors.)

then the ant's weight would be x^3 times bigger, but its legs would be only x^2 times thicker, so they would buckle.
Pressure on the ground = $\frac{\textbf{weight}}{\textbf{surface area}}$.

2.10.10 Design a bucket in the shape of a truncated cone that has a volume (capacity) of 9 litres.

This is a standard 2 gallon bucket.

If the radius at the bottom of the bucket is r, and the radius at the top is a ($r < a$), and the height of the bucket is h, then the volume V is given by $V = \frac{1}{3}\pi h(r^2 + ar + a^2)$.

Using values $r = 9$ cm, $a = 12$ cm and $h = 26$ cm gives V just over 9000 cm^3, so this would hold 9 litres. (Many other possibilities.)

Notice in this formula that if $a = r$ we obtain $V = \pi r^2 h$, the formula for the volume of a cylinder, and if $a = 0$ we obtain $V = \frac{1}{3}\pi r^2 h$, the formula for the volume of a cone, as we should.

Why do you think buckets are not usually cylinders?

Truncated cones will stack inside one another, are stable and are easy to reach inside and clean.

2.10.11 Archimedes (287-212 BC) said that if you put a sphere inside the smallest cylinder that it will just fit into, the volume of the sphere is $\frac{2}{3}$ of the volume of the cylinder.
Can you prove that he was right?

Answers:
Let r be the radius of the sphere. Then the height of the cylinder will be $2r$, so
volume of cylinder = $\pi r^2 \times 2r$ = $2\pi r^3$ and
volume of sphere = $\frac{4}{3}\pi r^3$ (standard result) which is $\frac{2}{3}$ of $2\pi r^3$.

What is the relationship between the two surface areas? (Assume that the cylinder has open ends.)

surface area of cylinder = $2\pi r \times 2r = 4\pi r^2$ and surface area of sphere = $4\pi r^2$ (standard result), so they're equal.

What if the ends are closed instead?

Then, surface area of cylinder = $4\pi r^2 + 2\pi r^2 = 6\pi r^2$; i.e., 50% more than the surface area of the sphere.

2.10.12 Archimedes' Principle.
Why do some things float and others don't?

Small insects and objects can sit on the surface of water because of surface tension, and that is a different phenomenon – they're not really "floating".

Whether something will float depends both on its mass (or weight) and on its shape.
As an object sinks into the water, the water pushes upwards on it and the force upwards is equal to the weight of the water the object has displaced. If the object can displace water with as much weight as the total weight of the object before it is completely submerged then it will float.

This will happen only if the average density of the object is less than the density of water (1 g/cm^3).

2.10.13 Inside Faces.
If you make a 3 × 4 × 5 cuboid from 1 × 1 × 1 cubes, how many faces of the cubes *can't* you see? (You're allowed to turn the cuboid around to look at it.)
Start with a 1 × 1 line of cubes and build up gradually.

Answer:
Imagine an $x \times y \times z$ cuboid where x, y and z are all positive integers.
Since each cube has 6 faces, altogether there are $6xyz$ faces. On the outside are $2xy$ visible faces from one pair of parallel faces, and $2xz$ and $2yz$ from the other two pairs of parallel faces. So the total number of inside faces must be $6xyz - 2xy - 2xz - 2yz$.

What if the cuboid is standing on a table, so that you can't see the faces underneath either?

In this case, say it's one of the $x \times y$ faces that is standing on the table. Then you just lose sight of xy faces, so the total number of unseeable faces increases to $6xyz - xy - 2xz - 2yz$.
If $y = z = 1$, then total = $3x - 2$, for example.

If it were one of the xz or yz faces that was standing on the table, then it would be the co-efficient of those terms that would change from –2 to –1.

2.10.14 **NEED** tape measures, possibly other things as well.

Answer: the value is not important; it's the process *adopted that matters.*

Estimate the volume of a human being.

Size will obviously depend on age of pupils.

Practical methods: could be done at home as a homework; e.g., mark side of bath before and after getting in (use something that will rub off!). Measure the difference in height and multiply by the cross-sectional area of the bath.
Theoretical methods: e.g., ignore hands, feet, etc., and treat the human body as a sphere on top of a cuboid with two identical cylindrical arms and two bigger identical cylindrical legs.

Theoretical approximation:
Head: $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi 10^3$ *= 4.2 litres;*
Trunk: $20 \times 50 \times 50 = 50$ *litres;*
Arms: $2 \times \pi r^2 l = 2 \times 3.14 \times 4^2 \times 50 = 5$ *litres;*
Legs: $2 \times \pi r^2 l = 2 \times 3.14 \times 6^2 \times 80 = 18$ *litres;*
So total estimate = 77 litres approx, which seems sensible.

(See similar task in section 2.2.17.)

If you were flattened by a steamroller so that you were only 5 mm thick, how big a splat would you make?!

Area = Volume/height = 0.08/0.005 = 16 m^2*; i.e., a 4 m by 4 m square!*
(Be cautious if some pupils may be upset by this!)

2.10.15 Find out the world record for the number of people who have simultaneously fitted inside a standard telephone box.
(Possible homework.)

Answer: About 20, depending on the type of telephone box and the exact rules about whether you have to close the door or be able to use the telephone!

Estimate a theoretical maximum.

The dimensions are about 3 ft × 3 ft × 8 ft, so the total volume = 72 cu ft (= 2 m^3 *approx).*
Assuming the average volume of a human being is 70 litres (see left), we would estimate a maximum of about 2000/70 = about 30 people. In practice a lot fewer.

Can estimate an average human volume (see above) or estimate by taking average density as 1 kg/litre (the same as water, since we just float) and an average human mass as 70 kg. So our volume is about 70 litres.

2.10.16 When no-one is using it, the water in the swimming pool comes up to 50 cm below the level of the floor outside the pool. How many people would have to get into the pool (completely submerged) to make it overflow?
(We'll assume the people are still, not jumping around and making waves!)

Answer: We could assume that the pool is 50 m by 25 m, so the area of the water's surface
= 50 × 25 = 1250 m^2*. We need to raise this by 50 cm, so the volume increase needed is*
1250 × 0.5 = 625 m^3*. If we take an average human volume as 70 litres, then it would take 625 000/70 = about 9000 people! (Not very practicable!)*

The bottom of the pool actually slopes from one end to the other so that one end is deeper than the other. What difference would it make if we took this into account?

It would make no difference since that extra space will be filled with water throughout.

2.10.17 How many identical packets (cuboids 3 cm × 4 cm × 5 cm) can you fit into these cuboid containers?

1. 30 cm × 40 cm × 50 cm;
2. 30 cm × 40 cm × 51 cm;
3. 30 cm × 40 cm × 52 cm;
4. 30 cm × 40 cm × 53 cm;
5. 30 cm × 40 cm × 54 cm.

Answers:
1. 1000; 2. 1020; 3. 1040, 4. 1040 (still), 5. 1080.
Provided the packets fill the entire container with no empty space, you can divide the volumes; i.e., for question 1, $\frac{30\times40\times50}{3\times4\times5} = 1000$ *, but a safer way (and necessary if there are going to be any gaps) is to think how many rows you'll get along each dimension; i.e.,* $\frac{30}{3} = 10$ *along the 30 cm side,* $\frac{40}{4} = 10$ *along the 40 cm side and* $\frac{50}{5} = 10$ *along the 50 cm side, and*
10 × 10 × 10 = 1000.

When the answers to these divisions are not integers, you always need to round down.

Try these ones (same size packets):
1. 10 cm × 15 cm × 20 cm;
2. 10 cm × 10 cm × 15 cm;
3. 10 cm × 10 cm × 10 cm;
4. 10 cm × 11 cm × 12 cm.

1. 50; 2. 20; 3. 12; 4. 18.

In general, if the sides of the container have lengths A *,* B *and* C *, and the sides of the packets have lengths* a *,* b *and* c *, you need to work out the six products* $\frac{A}{a}\frac{B}{b}\frac{C}{c}$ *,* $\frac{A}{a}\frac{B}{c}\frac{C}{b}$ *,* $\frac{A}{b}\frac{B}{a}\frac{C}{c}$ *,* $\frac{A}{b}\frac{B}{c}\frac{C}{a}$ *,* Try making up some puzzles like these.

A spreadsheet makes this much easier.

2.10.18 How long is a toilet roll?
You want to know how much paper there is on a toilet roll without unrolling the whole thing. What measurements could you take?

Another way of arriving at this formula is to think about the area of the end of the roll (the cross-sectional area), which is $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$ *, and this will be the same as the thickness of one sheet multiplied by the length of the whole roll.*

So length of the roll is $\frac{\pi(R^2 - r^2)}{t}$ *again.*

$\frac{A}{c}\ \frac{B}{a}\ \frac{C}{b}$ *, and* $\frac{A}{c}\ \frac{B}{b}\ \frac{C}{a}$ *, in each case doing "integer division" (normal division but rounding down and discarding the remainder). You have to see which of these six products gives the maximum number of packets.*

Answers:
One option would be to weigh the roll and to weigh one sheet and do a division. (It would be more accurate to count off 20 sheets, say, to weigh and then divide by 20.) To do this, you would need an accurate balance and you would have to weigh a cardboard tube separately and subtract this from the total. You would calculate how many sheets were on the roll and then multiply this by the length of one sheet.

A second option would be to measure the thickness *of one sheet (again, you would measure 20, say, and divide by 20) and the thickness of the roll, and divide to find out how many layers there are on the roll. This number can be multiplied by the average circumference* $\frac{1}{2}(R + r)$ *, where* R *is the outer radius and* r *is the radius of the cardboard tube. Since the thickness of paper on the roll is* $(R - r)$ *, and if* t *is the thickness of one sheet, then the number of layers on the roll is* $\frac{R-r}{t}$ *so the length of the roll is* $\frac{R-r}{t} \times 2\pi\left(\frac{R+r}{2}\right)$

$= \frac{\pi(R^2 - r^2)}{t}$.