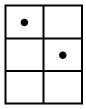
3.4 Combinations

- "Combinations" refers to the number of ways that things can happen or be done.
  "Permutations" includes all the different orders that those combinations could come in.
- "Pascal's Triangle" is sometimes called the "Chinese Triangle", because it pre-dates Blaise Pascal (1623-1662) by hundreds of years.

3.4.1 Braille.
Braille is an arrangement of between 0 and 6 raised dots in a 2 × 3 rectangle;
e.g., the letter E is



(the lines of the rectangle are not normally drawn in).

How many other possible arrangements can you make just using 2 dots?
How many arrangements can you make if you can use any number of dots from 0 to 6.

(Every square can be either a "dot" or "no-dot" – that explains why $d = 2$ and $d = 5$ have the same number of arrangements – their arrangements are like "negatives" of each other.)

Are there enough different arrangements to cover the alphabet, digits and necessary symbols like full stop?

It would be good to have some Braille books to look at and to try to decipher (using a Braille alphabet – available, for example, on the internet).

Answers:

number of dots	***number of arrangements***
0	*1*
1	*6*
2	*15*
3	*20*
4	*15*
5	*6*
6	*1*

If the number of dots is d, then the number of arrangements is ${}^{6}C_{d}$. These numbers are the seventh line of Pascal's triangle.

$${}^{6}C_{d} = \frac{6!}{(6-d)!d!}$$

The total number of arrangements is $2^6 = 64$, because there are two possibilities for each square and there are six squares.

This is plenty of characters.

3.4.2 Investigating Pascal's Triangle.
Write out Pascal's Triangle as far as the 7th or 8th row at least (see sheet).

Look for these types of numbers and describe where they come in the triangle.

1. lines of "1"'s;
2. natural numbers;
3. triangle numbers;
4. tetrahedron numbers;
5. powers of 2;
6. powers of 11;
7. ${}^{n}C_{r}$

What other patterns can you find?

(See the sheet for further details.)

Answers:

1. *diagonally down each side;*
2. *diagonal lines next to the diagonals of 1's;*
3. *the next diagonal lines to the natural numbers;*
4. *the next diagonal lines to the triangle numbers;*
5. *the sums of each horizontal row;*
6. *the* digits *along each row up to and including the 5th row;*
7. *the $(r+1)^{th}$ number (or the $(n-r)^{th}$ number) along in the $(n+1)^{th}$ row.*

If you shade in the squares containing odd numbers and leave the other squares white, you get the so-called Sierpinski Triangle (1882-1969). This drawing is a fractal *(when you zoom out it looks the same).*

3.4.3 Why must there be at least two people in the world with the same number of hairs on their heads?

It's obvious that if you have two completely bald people, then they have the same number (0) of hairs on their heads.

Answer: There are more people in the world than there are hairs on anyone's head, so everyone can't have a different number of hairs (pigeonhole principle)!

Some pupils may question this assumption, but 6.5 ×

10^9 hairs would be a very hairy person!

3.4.4 **NEED** Squared paper, preferably at least as big as a 2 cm × 2 cm squared grid (see sheet).
Routes through a Grid.
If you're going from one place to somewhere else in a city, then there may be more than one way of getting there.
Generally, that's complicated, but it's simpler in New York (or Milton Keynes) because of the way the streets are laid out.

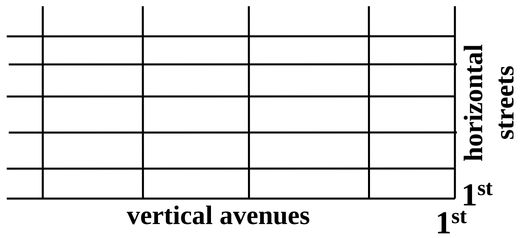
Does anyone know what's special about the layout of the streets in New York?

Roads are laid out so that they mostly cross one another at 90° (see right). North-South roads are called avenues, *and East-West roads are called* streets*. They're numbered from 1st (south-east corner). (So a junction can be described as "5th avenue, 42nd street", for instance.)*

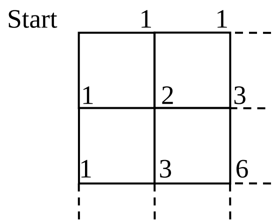
Investigate routes through a simplified version. Write on the diagram the number of ways of getting from the start (top left corner) to each crossroads. You are only allowed to go right and down, because we want routes that are as short as possible (never go back towards the start).

Can you explain your results?

Once you see what is going on, try to complete the pattern of numbers.



The avenues are split into "short blocks" (about $\frac{1}{20}$ mile); the streets into "long blocks" (about $\frac{1}{4}$ mile).
Milton Keynes uses a system based on H (horizontal) and V (vertical).



Pascal's triangle. (We have to say that there is "1 way" of getting to the "start".)

Wherever you are you must have come via the position immediately above or the position immediately to the left. So the number of ways of reaching any position is the sum of the numbers of ways of reaching these two positions. That generates Pascal's Triangle.

3.4.5 I write 6 letters and address 6 envelopes. I put each letter into an envelope. How many ways are there of getting every letter in the *wrong* envelope?

This is called the number of "derangements".

It turns out that $!n = \left[\frac{n!}{e}\right]$*, where the square brackets indicate that you round the answer to the nearest integer.*

Answer: Very hard!

$$6!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right) = 265 \text{ ways}$$

This is called "6 sub-factorial" and written $!6$ *(instead of* $6!$*).*

In general, $!n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}$

3.4.6 How many ways are there of arranging the letters in your name. (The rearrangements don't have to make real words!)
e.g., for GEORGE it is $\frac{6!}{2!2!} = 180$ *(because of the two G's and two E's).*

Answer: If the name contains a *letters, of which* b *are the same as each other,* c *are the same as each other, etc., then the number of arrangements is* $\frac{a!}{b!c!...}$*;*

3.4.7 Find out what "entropy" is and why it is important in Science.

This is called the "2nd Law of Thermodynamics".

Answer: Entropy is a measure of how "disordered" a system is; the total entropy of the universe always increases when anything happens.

Pascal's Triangle and ^nC_r Values

n value \ r value	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

^nC_r is the number of ways (regardless of the order) of choosing *r* things from *n* things.

$$^nC_r = \frac{n!}{(n-r)!r!} \text{ and } 0! = 1$$

- $^nC_0$ and ^nC_n are always 1 (there's one way of choosing no objects, and one way of choosing the whole lot).
- ^nC_r and $^nC_{n-r}$ are always the same as each other (if you choose *r* objects, you've inevitably chosen (*n* – *r*) objects, because they're the ones you've left behind).
- $^nC_1$ and $^nC_{n-1}$ are always *n* (there are *n* ways of choosing just 1 [it could be any of them] and *n* ways of leaving just 1 behind [again, it could be any of them]).
- From the table above, you can see that $^{n-1}C_{r-1} + {}^{n-1}C_r = {}^nC_r$, and

$$\frac{(n-1)!}{(n-1-(r-1))!(r-1)!} + \frac{(n-1)!}{(n-1-r)!r!}$$

$$= \frac{(n-1)!r}{(n-r)!r!} + \frac{(n-1)!(n-r)}{(n-r)!r!}$$

$$= \frac{(n-1)!(r+(n-r))}{(n-r)!r!} = \frac{(n-1)!n}{(n-r)!r!} = \frac{n!}{(n-r)!r!}$$

To understand this, think about 1 of the *n* objects separately from the other (*n* – *1*) objects.
This one is either included in your choice of objects or it isn't.

If it is, then there are (*r* – *1*) objects left to choose from (*n* – 1) [$^{n-1}C_{r-1}$ ways of doing that].

On the other hand, if that object *isn't* included, then you have to choose all *r* objects from only (*n* – 1) objects, and there are $^{n-1}C_r$ ways of doing that.

These two possibilities we've been thinking about are *mutually exclusive*, so the total number of ways of choosing *r* objects from *n* will be the sum of these two numbers;

i.e., $^{n-1}C_{r-1} + {}^{n-1}C_r = {}^nC_r$.

If you can follow this, you're doing well!

Start

Start
