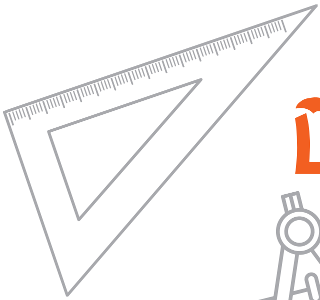
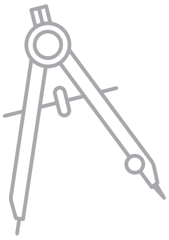
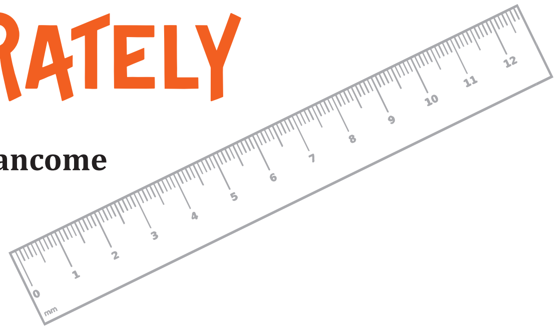




DIAGRAMS NOT DRAWN ACCURATELY



By **Colin Foster and Tom Francome**

When doing past examination questions, students occasionally complain about the 'Diagram not accurately drawn' statement that sometimes appears next to geometrical figures: "Why not? Couldn't they be bothered? Why didn't they make some effort and do it properly?" The answer to this is usually that the examiner didn't want to make the question answerable by measurement. The question is attempting to test something like calculation of angles, and so, if the angle can be measured with a protractor, this would provide an alternative method that the examiner wishes to block. Of course, any method relying on measurement could only ever be approximate, but a student might use it to confirm or refute a calculation that they have done, or they might assume that an angle that seems to be near, say 30°, *is* 30°, and so it could provide help that the examiner doesn't wish to offer. But what about in the classroom? Practices used in high-stakes assessments are often poor guides to what is likely to be most helpful during teaching. When should mathematical diagrams be drawn accurately and when should they, perhaps deliberately, be '*not* to scale' (see Note)?

When drawing by hand on a whiteboard, some mathematics teachers pride themselves on their skilful use of board rulers and other giant-sized geometrical tools, and nowadays, with so much mathematics in classrooms being drawn with technology, it's no longer difficult to make drawings look precise. A straight line can look straight, right angles can be precise, and a circle can look very circular – at least if the aspect ratio settings on the projector are suitably set (see Francome, 2016)! The teacher might regard this as simply 'having high standards of presentation', and might expect students to also always draw straight lines with a ruler and circles or arcs using compasses, so that their books look neat.

But is this a good stance to take? Although measurement can be very useful for getting a sense of relationships and properties, ultimately mathematics is not a measurement science or a branch of technical drawing, but a discipline that relies on deduction. We *deduce* that an angle must be 30°, even if it actually looks more like 45° in our sketch – this is the distinction between an accurate drawing and a sketch – and the mismatch between the two shouldn't disturb us. For this reason, other mathematics teachers *discourage* students from using rulers in any topic except things like scale drawing and loci and constructions, because they want them to shift from relying on appearance and measurement and move towards trusting only what can be honestly deduced from the given information. For this kind of teacher, if you tell them that their straight lines look crooked or curvy, and their circles look like ovals, they take it as a compliment.

Imagine a student moving from one of these teachers to the other – how confusing! It would seem to be one of those areas where some departmental consistency would be useful. But where is a sensible position to settle on between these extremes? If a worksheet of questions on Pythagoras' Theorem consists of exactly the same right-angled triangle copied and pasted throughout (perhaps in different orientations), with different lengths written on the sides, is this just laziness on the part of the designer, who couldn't be bothered to draw a new triangle for each question? Or is it perhaps a useful abstraction, in which the triangle figure has become almost a symbol for 'any right-angled triangle' (as writing Δ in Δ ABC can refer to *any* triangle), so foregrounding calculation rather than risking being misled by the visual impression? Perhaps this is an important way of focusing attention on the numbers rather than on the superficial appearance.

Sometimes textbooks seem confused about the distinction between an accurate drawing and a sketch, and present drawings of various shapes and ask students things like, 'Which ones are a square?', or 'Which ones have two lines of symmetry?' Students are supposed to do this just by looking, which seems to suggest that 'a shape is a square if it looks like one', or by measurement, which can only ever be approximate. We suppose you might be able to say by inspection or measurement that a shape was definitely *not* a square, but you can never deduce that something *is* a square just by looking or measuring. If you zoom in far enough, on even the most high-resolution image, the edges will eventually cease to look straight and the vertices will cease to appear point-like. No one ever sees a true square in the real world – perfect squares are mental objects. So, the question for the student becomes, ‘Do you think this square is square enough that it is likely that the authors intended it to be considered a square?’

We’ve seen questions where the textbook has intended to offer something like a 5.1 cm by 4.9 cm rectangle, and students are supposed to decide by measurement that it *isn’t* a square, and if they say that it is they are accused of not being accurate enough. This seems unreasonable, given that no degree of tolerance is specified in the question, and given that often quite inaccurately-drawn shapes appear in mathematics textbooks without comment (see Foster, 2012). We think questions like ‘Is it a square?’ only really make sense in scenarios where the exact properties of the shape are stated, or where the construction of the shape is clear (such as by classical construction), or where the shape is drawn with its vertices on the lattice points of a grid. Other ways to do this are for the shape to be made by juxtaposing previously-well-defined shapes, or by paperfolding, where the symmetries produced are clear from how they are made.

We think the best position to take on accuracy of drawings is that what is preferable depends on the didactical purpose in the particular situation. When concepts are being introduced, and where we are seeking to build students’ intuitions, accurate drawing can be a powerful scaffold to enable them to make sense of what’s going on. So, in learning about the triangle inequality, for instance, it can be very valuable to have students accurately construct (or try to) triangles with sides such as 3 cm, 4 cm and 5 cm, and also with sides such as 3 cm, 4 cm and 10 cm, to see why the latter can’t exist. But this is not saying that a triangle exists if you think you can draw it. (We have seen students insist that they can draw a 3 cm, 4 cm, 7.1 cm triangle.) The endpoint is that we want students to be able to abstract, and deduce things from three given lengths, without even a sketch drawing.

Sometimes we will even want to present students with *misleading* diagrams. For example, we might ask them ‘Can you work out the area in Figure 1?’ Klymchuk (2015) found that, of the 76 teachers who answered this question, only six noticed that this was impossible – and this was in a situation where they were informed in advance that some of the questions that they would be given had a ‘catch’. Sometimes it can be difficult to decide whether something is deliberately misleading or not. Using different scales on the horizontal and vertical axes of a graph is often necessary if the quantities have very different-sized units (e.g., plotting $y = \sin x$, where x is in degrees), but it means that the angles and gradients are ‘wrong’. But using equal scales on the axes would prevent one period of the graph being viewable on any reasonable-sized piece of paper.

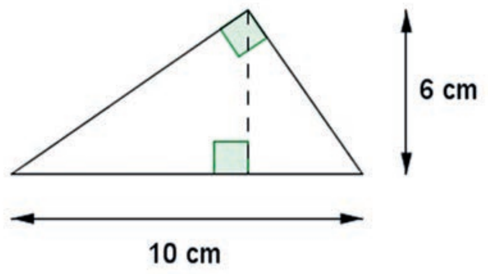


Figure 1: Adapted from Klymchuk (2015)

In general, the issue is the extent to which we want the presentation of the diagram to ‘do some of the work’ for the student. Parallel lines drawn so that they *look* parallel mean that students might be able to ‘guess’ correct things, such as corresponding angles being equal, and this can be very helpful when *introducing* these concepts. But, ultimately, we want students’ knowledge to be robust enough to operate even when the lines are deliberately drawn so that they look *non*parallel, and any “parallelness” is communicated by arrows included on the lines, or by a statement that the lines are parallel, so that students deduce relationships directly from the given properties. The challenge is how we assist learners in deciding what matters in any particular situation.

We need to work towards the fluency that enables students to work with things like empty number lines, to do calculations like

$$3125 - 7.2 + 2.0001,$$

without stressing about trying to draw it to scale. We want them to be able to calculate the missing angles in a triangle when one angle is 0.5° (e.g., in scenarios such as estimating the diameter of the sun from the earth), without feeling like they are supposed to try to draw an accurate representation of the situation. The default should be that we assume the diagram is *a sketch*, not an accurate drawing.

However, we also need students to develop a good approximate sense of size, so that they become familiar with what is a realistic magnitude. If you ask students to shut their eyes and indicate with their fingers in the air how big they think 10 cm is, the responses can vary wildly. Similarly, if you ask students to stand up, shut their eyes and rotate themselves clockwise by 10°, again when they open their eyes they have usually rotated by very different amounts, and are all facing in quite different directions. Perhaps, if students are constantly presented with lengths and angles on the page or the board which bear no relation to the numbers written beside them, we train out of them any realistic appreciation of magnitude?

We think that accurate diagrams can be an important scaffold, particularly when encountering new concepts, but the default should be ‘never trust the magnitudes in drawings’. Ultimately, students need to be presented with a variety of diagrams, some of which may even cross the line into 'misleading', so that we eventually wean them off relying on visual appearance. We want students to become sufficiently mathematically aware to make sensible decisions about what can be taken for granted and what can't in different situations. Diagrams always require some mathematical thinking.

Note

'Drawn accurately' and 'drawn to scale' are perhaps not quite the same thing. A right-angled triangle, for instance, might have its right angle drawn accurately but its sides might not correspond to the lengths written beside them. It is quite subtle which features of a geometrical diagram it is acceptable to make assumptions about and which it is not. We often have to assume that lines that look straight are – if we didn't do that, almost all geometry problems would be impossible. We assume that points that look coincident are, otherwise they would be drawn to make them look distinct. Other cases are less clear-cut. In some contexts, we might assume that a point that appears midway between two others is the midpoint. But should we assume that angles that look like right angles are, if they are not specifically marked as such? Can we assume that, if we were *not* supposed to assume that, then they would be deliberately drawn to be obviously *not* right-angled? It can be difficult with diagrams involving tangents to circles, for example, to avoid the impression that the tangent and radius are at right angles, because that would entail making the circle non-circular, so this property can be difficult to obscure. When a perpendicular height is marked onto a figure (e.g., for a question about calculating the area of a non-right-angled triangle, like in Figure 1), it isn't always explicitly stated that that measurement is perpendicular to a side – we just assume it is, because it's conventional to include that measurement, and because it is what is needed to solve the problem.

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