make a rotation of 180°' seemed better. These claims then opened inquiry such as, 'Do any two reflections compose to a rotation?' that gave space for claims specifying precise results of composition *e.g.,* 'Reflections about lines forming a D° angle compose to a rotation of $2D$° (depending on how you take the composition)'. Students transformed prompts into sites for generating and refining claims (compare Boero, Garuti, Lemut & Mariotti, 1996).

To support the decompression of the definition of mathematical claim, Lai set up activities where students learned to ask mathematical questions about a situation, judged a collection of given claims for a given question, and discussed how claims may be 'satisfying'. Over several years, students consistently now say that 'satisfying' means accounting for as many possibilities of examples as possible, as unambiguous as possible, and ideally mathematically true. The proposed decompression of mathematical claim synthesizes these students' discussions on the notion of 'claim' and the authors' collective experiences as mathematician, teachers, and scholars.

We focused here on mathematical claim because of its special place in a mathematical agenda. Claims determine mathematical activity, including its tools. They communicate and organize knowledge along the way to further discovery (Hanna & Barbeau, 2008).

We illustrated here that decompressing mathematical claims to be accessible, descriptive, normative, and participatory can be a conceptual tool for designing teaching to cultivate learners' mathematical dispositions. What about another process, such as proof? We know from the literature that proof may be an *a priori* justification (Hanna & Barbeau, 2008); or a transparent justification, where community members can fill in any gaps given sufficient time and motivation; or a perspicuous justification that provides understanding (Czocher & Weber, 2020). However true these aspects, they do not capture the role of the community in creating, writing, judging, and certifying proof.

Full participants of an epistemic community shape the norms of that community. Decompressions are a conceptual tool for students and instructors to develop shared language for the processes governing mathematical inquiry and argumentation, and their organization. When a classroom community—including students and instructor—can meaningfully discuss worthwhile norms for knowledge processes, they shape the norms of that class, and they have the potential to shape the norms of future classes where they find themselves. As educators and researchers, we hope this essay on the definition of mathematical claims promotes discussion around definitions of mathematical knowledge processes as objects, the role of social context (such as proximal learning communities), and the compressed nature of mathematics as a discipline as well as the implications for compression and decompression for teaching and learning mathematical processes.

Acknowledgements

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Note

[1] We adopt Engle's (2011) definition of 'intellectual agency' that learners "are 'authorized' to share what they actually think about the problem in focus rather than feeling the need to come up with a response that they may or may not believe in, but that matches what some other authority like a teacher or textbook would say is correct" (p. 8).

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Priority of operations: necessary or arbitrary?

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We reflect here on the priority of operations, where an expression such as $3 + 4 \times 5$ is taken to be equal to $3 + 20 = 23$, rather than to $7 \times 5 = 35$. In this example, the multiplication takes priority over the addition, meaning that it is carried out first, even though, when reading left to right, the addition operation is encountered first. In order to teach the priority of operations, we see it as important to establish whether it is 'merely' a common convention—something that could be otherwise—or whether it follows mathematically and 'can be proven'. We were surprised to discover that the mathematics education literature appears to be unclear on this point.

Hewitt (1999) introduced the terms *arbitrary* and *necessary* knowledge, where arbitrary knowledge consists of names and conventions which could be otherwise, whereas *necessary* knowledge consists of statements that *must be so* mathematically. 'Telling' learners arbitrary things may be appropriate, but we generally seek to avoid telling necessary knowledge; instead, we offer tasks that aim to assist learners in becoming aware of such things, without being told explicitly. The arbitrary may be important to be learned (memorised), but the necessary is where mathematics lies. This is because it is only with the necessary that we can justify and prove properties and relationships (Hewitt, 1999).

Claims that priority of operations is necessary knowledge

It has been claimed in the literature that the priority of operations is mathematically necessary (Bay-Williams & Martinie, 2015; Mattock, 2019; Zazkis & Rouleau, 2018). Indeed, the idea that the priority of operations is an arbitrary convention has been described as a "myth" (Bay-Williams & Martinie, 2015, p. 22). For example, Mattock (2019) claims that, because of the commutativity of addition, an expression such as $3 + 4 \times 5$ must be equal to $4 \times 5 + 3$. Since the first expression is either equal to $3 + 20 = 23$ or $7 \times 5 = 35$, and the second expression is either equal to $20 + 3 = 23$ or $4 \times 8 = 32$, the only way in which these two expressions can be equal is if they are both equal to 23, and this tells us that multiplication must take precedence over addition.

Zazkis and Rouleau (2018) offered a related argument that the priority of operations is necessary:

> While this convention may appear as an arbitrary decision of mathematics, it is actually a necessary result of interpreting multiplication as repeated addition ... Consider for example $2 + 5 + 5 + 5 + 5$ vs. $2 + 4 \times 5$. Obviously, $5 + 5 + 5 + 5$ can be rewritten as 4×5. In order to assure [sic] that both expressions lead to the same result, multiplication should be performed before addition. (p. 144)

Similar arguments have been used to explain why 2×5 ^ 3 must be equal to $2 \times 5 \times 5 \times 5$, and cannot be interpreted as (2×5) ^ 3 (Bay-Williams & Martinie, 2015, p. 22). Authors have claimed that "the distributive property implies a natural hierarchy" [1] suggesting that, because multiplication is, in some sense, 'more powerful' than addition, just as exponentiation is 'more powerful' than multiplication, the priority of operations *must* respect this order.

Priority of operations as arbitrary knowledge

We think that there is a circularity to these arguments, and that they unintentionally assume what they are trying to show. Our familiarity with the conventional priority of operations can make it hard to see when this knowledge is being unconsciously smuggled into the argument. By operationalising the commutativity of addition as $3 + 4 \times 5 = 4 \times 5 + 3$, Mattock's (2019) argument already assumes that multiplication takes precedence, since the 4×5 is treated as a fixed unit. If the author did not already know the conventional priority, they might instead operationalise the commutativity of addition by writing $3 + 4 \times 5 = 4 + 3 \times 5$ (simply switching the 3 and the 4 that are either side of the addition symbol). This parallels what we would do if we were applying the commutativity of *multiplication* in an expression such as $3 \times 4 + 5$ and equating it to $4 \times 3 + 5$. There would seem to be no *a priori* way to know which of these is the appropriate application of commutativity of addition, unless the conventional priority of operations is already known. The expressions $3 + 4 \times 5$, $4 + 3 \times 5$, $5 \times 4 + 3$ and $5 \times 3 + 4$ would all be equal to 7×5 (or 5×7) if addition took precedence over multiplication (opposite to the usual convention). Note that the latter two need addition to take *priority*, whereas the former two just follow the left to right *order*. Treating + and × merely as binary operations, without any prior knowledge of their relative priority, we would be completely unable to determine that either operation had priority over the other. A similar situation occurs if we begin with the commutativity of *multiplication*, rather than addition. Someone without knowledge of our cultural choices would be unable to decide one way or the other.

In a similar way, Zazkis and Rouleau's (2018) comparison of $2 + 5 + 5 + 5 + 5$ and $2 + 4 \times 5$ takes for granted that 4×5 is a fixed unit. $2 + 5 + 5 + 5 + 5$ is certainly $2 + (4 \times 5)$, but it is precisely the removal of the brackets that the conventional priority of operations allows. The expression $2 + 4 + 2 + 4 + 2 + 4 + 2 + 4 + 2 + 4$ can also be seen as a repeated addition, of $2 + 4$, which can be similarly grouped as $(2 + 4) \times 5$.

One way to attempt to quieten the 'curse of knowledge' that biases us towards the conventionally correct result is to replace + and × with arbitrary symbols. Using ◇ and ● to represent two unspecified commutative binary operations, we do not think it is possible for anyone to say which of these is 'correct':

$$3 \diamond 4 \bullet 5 = 3 \diamond (4 \bullet 5)$$
$$3 \diamond 4 \bullet 5 = (3 \diamond 4) \bullet 5$$

If ◇ is addition and ● is multiplication, then, under the conventional priority of operations, the first equation is true and the second one is false. If ◇ is multiplication and ● is addition, then the first equation is false and the second one true. But if we do not know which symbol is which then we cannot say which is correct.

In Figure 1, we have made the expressions unambiguous by employing brackets, and we have illustrated the priority using trees [2]. There can be no doubt that the first expression is equal to 35 and the second is equal to 23. We see the question of priority of operations as being about which one of these we *conventionally* wish to mean when we write the expression as $3 + 4 \times 5$ *without brackets*. A convention of working left to right would in this case lead to addition happening first; this can be overruled only if a convention of prioritising multiplication over addition is in operation. This

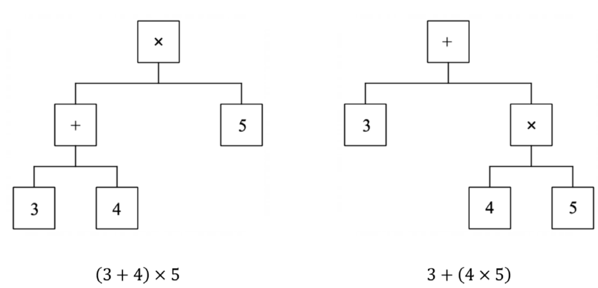

Figure 1. Using trees to represent different priorities of operations.

seems to us to be an arbitrary notational choice. It is not so much about what operation 'must' happen first as about how we decide to interpret potentially ambiguous notation unambiguously.

Pragmatic choices

Ambiguity in these kinds of expressions can always be resolved by inserting brackets, given an overriding convention to prioritise brackets above all else. However, these additional brackets require extra effort by taking up space, both on the page and in mental processing. It may be regarded as more elegant to avoid using unnecessary brackets, but 'if in doubt put them in' might be advisable. This observation could lead us to seek a priority of operations convention that minimises the frequency with which brackets must be used in the kinds of expressions we expect to be writing most often.

The ease with which we can write the sum of two numbers in standard form, such as $3 \times 10^5 + 2 \times 10^4$, derives from the *implicit brackets* around the products and the powers, which would be cumbersome to write out fully as (3 × (10 ^ 5)) + (2 × (10 ^ 4)). A similar argument applies for writing polynomials such as $5x^2 - 3x + 2$. However, we note that the superscript notation for indices *presupposes* a certain priority, because 10^{5+1} is transparently different from $10^5 + 1$. A more neutral way to write these expressions, that leaves open the issue of priority, is as 10 ^ 5 + 1, which is why we have used the ^ notation several times in this article. The conventional choice of priority of operations is internally consistent in the sense of a hierarchy of binding power, such that + < × < ^. If we had instead, for example, + < ^ < ×, so that $1 + 2x^3 = 1 + (2x)^3$, this would undoubtedly be a worse choice.

If the priority of operations is truly arbitrary, it should be possible to alter—even reverse—it without 'breaking mathematics'. In other words, messing about with the conventional priority may be inconvenient and inefficient, but it should not lead to any mathematical contradictions. There is a website [3] which contains a calculator which allows the user to vary the priority of operations and see what happens. For example, if instead of BIDMAS (Brackets-Indices-Division-Multiplication-Addition-Subtraction), we reverse this, to obtain SAMDIB, we would write calculations such as 4 × 5 + 3 = 4 × 8 = 32. The fact that it seems perfectly possible to do this convinces us that the priority of operations is indeed mathematically arbitrary, although, like most arbitrary choices in mathematics, this does not imply that the choice was made at random or on a whim. The convention is convenient and sensible, because the written symbolic notation tends to have numbers and letters visually closer together for the prioritised operations (e.g. $2a + 3$ has the '2' and the 'a' closer together than the 'a' and the '3'). However, it is not *mathematically* necessary. The only thing that would change with alternative choices would be the set of situations in which we would *need* to use brackets and how convenient any alternative convention might or might not be, given the kinds of expressions most commonly written.

Our discussions about this have led us to conclude that the priority of operations is indeed an arbitrary convention. However, does this mean that we simply 'tell' it to learners and move on, merely allowing time for them to just 'get used to it'? We think not. We think it is important for learners to be aware of when something is a choice and when it is not. When something is arbitrary, it is helpful to experience alternatives, so that it is clear that the convention is a choice. With priority of operations, we might ask learners to consider what an expression such as 3 + 4 × 5 is, or 'might be', equal to, hoping that some learners will offer 23 and others 35. The fact that there are potentially different possible answers raises the need for a convention. Then, since this is a socially-agreed convention, which is arbitrary, learners need to be informed of what it is. The emphasis at this point is on assisting memory so that the convention becomes an almost automatic way of viewing an expression (Hewitt, 1999).

Notes

[1] See 'Order of operations: historical caveats' at https://tinyurl.com/FLM-44-2-3

[2] See https://tinyurl.com/FLM-44-2-4 for a convenient tree notation calculator.

[3] At https://tinyurl.com/FLM-44-2-5

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On the necessity of order and the field axioms

RINA ZAZKIS

In 'Priority of operations: necessary or arbitrary?' (in this issue) Foster, Francome, Shore, Hewitt and Sangwin argue that the conventional order of operations in arithmetic is an *arbitrary* convention. That is, it is not a *necessary* property, using Hewitt's (1999) distinction between arbitrary and necessary. To advance their case, the authors: (1) claim that the arguments provided to support the position that the order is necessary are flawed, and (2) introduce an alternative arithmetic in which the priority of operations is administered in discord with conventional mathematics. Later, they also (3) attempt to explain why the convention is a 'pragmatic' choice.

There are several major problems with the provided arguments and examples; I address each one in turn.

On the unsubstantiated accusation of a flaw

Foster *et al.* cite several examples from literature in which it is argued that the conventional order of operations in arithmetic (in particular, the priority of multiplication over addition) is necessary. The authors suggest that these arguments are based on a "curse of knowledge" (p. 25); as a