WHAT IS A FRACTION?

by **Colin Foster, Tom Francome, Dave Hewitt & Chris Shore**

Mathematics is often portrayed as a 'precise' subject, in which everything must be carefully defined. But this is much easier for some mathematical concepts than for others. It may be important to nail down exactly what we mean by something like a 'trapezium' (Note 1), but what about a 'point' or a 'line'? Although (perhaps because?) these terms are familiar, they can be as hard to define as many everyday words are (try defining a 'cat', for example). For some mathematical concepts, it is very important for students to move beyond saying 'I know it when I see it'. For example, to be a square, a shape must possess precisely the right properties – the claim "It's a square because it looks like one" is not a mathematical argument. Even a slight deviation from equal side lengths or equal angles – or a microscopically tiny 'fifth side' – will rule it out. Conversely, non-standard examples of squares (e.g., a square that is tilted) are 'in', even if they might look odd to students, so long as they satisfy the definition. So precision seems really important here.

But not all mathematical words seem to be like this. Take 'fraction', for example. Whereas 'rational number' has a precise definition – any number which can be written as $\frac{a}{b}$, where a and b are integers and $b \neq 0$ (Note 2) – 'fraction' seems much harder to pin down. For instance, which of the things in the box below do you consider to be 'fractions'?

$\frac{4}{6}$	$\frac{6}{4}$	$1\frac{2}{3}$	$\frac{0}{3}$	$\frac{3}{0}$	$\frac{0}{0}$	$2 \div 3$	$2/3$
three sevenths	$\frac{6}{3}$	$\frac{1}{1.5}$	$\frac{1}{\left(\frac{3}{2}\right)}$	$\frac{\left(\frac{2}{3}\right)}{5}$	$\frac{1}{1+\frac{1}{2}}$	$\frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}$	3^{-1}
$\left(\frac{3}{2}\right)^{-1}$	$\frac{-2}{3}$	$\frac{-2}{-3}$	$-\frac{2}{3}$	0.5	$\frac{x}{6}$	$\frac{6}{x}$	$\frac{\pi}{6}$
metre per second (m/s)		$\frac{\sqrt{2}}{3}$	$\frac{3}{\sqrt{2}}$	$\frac{x}{2x-1}$	$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$	$\frac{dy}{dx}$	$\frac{\Delta y}{\Delta x}$

How should we decide? To create many of them, we had to use the 'fraction' template in *Equation Editor* in *Microsoft Word*, but that is surely no reason to say that they are *mathematically* 'fractions'. Perhaps we might look to etymology for help. The word 'fraction' comes from the Latin 'to break into parts' (think of *fracture*, *refract*, *fragment*). So, perhaps we should say that $\frac{1}{6}$ is a fraction of 1 and $\frac{\pi}{6}$ is a fraction of π? (This might suggest that the denominator of a fraction should always be an integer (or at least rational), whereas the numerator might not need to be.) Maybe the response to "Is it a fraction?" should be "A fraction of what?"

Let's consider a definition from the stricter end of the spectrum of possibilities. Someone might say that a fraction must have *integer* numerator and *non-zero integer* denominator. That's a definition that has the advantage of being clear and simple to state – let's call it the 'integer' definition. With a definition like that, we can immediately decide 'yes' or 'no' for many of the examples above. But not for all. For example, with this definition, $\frac{x}{6}$ may or may not be a fraction, depending on the value of x. So, the left-hand side of the equation

$$\frac{x}{6} = \frac{1}{4}$$

is *not* a fraction (because x is not an integer), whereas the left-hand side of the equation

$$\frac{x}{6} = \frac{1}{2}$$

is a fraction, because here x *is* an integer. This means that the first equation is of the form 'non-fraction = fraction', whereas the second is 'fraction = fraction'.

Is this a useful distinction to make for any purpose? Wouldn't we want students to think of these two equations in essentially the same way, as both being of the form 'fraction = fraction' – and then we can use everything we know about fractions to solve them? Whether x turns out to be an integer seems beside the point, and there seems to be nothing to be gained by calling one a fraction and not the other.

To consider a different scenario, if students are converting, say, $\frac{\pi}{3}$ to $\frac{2\pi}{6}$, we might wish them to see that as essentially the same process as converting $\frac{1}{3}$ to $\frac{2}{6}$, even though the first pair, by this 'integer' definition, are *not* fractions and the second pair *are*. The fact that the first two are irrational seems irrelevant here – and, indeed, the issue isn't really about rational versus irrational anyway, since something like $\frac{2\pi}{3\pi}$, by this definition, would be *rational* but *not* a fraction – although it simplifies to a fraction. The fact that π happens not to be an integer feels beside the point in all of this.

As we have seen, this 'integer' definition of a fraction means that so-called 'algebraic fractions', like $\frac{x}{3}$, are not necessarily fractions. To take a trickier example, $\frac{x}{2x-1}$ *is* a fraction if x is an integer, but is it a fraction if x is *not* an integer? This is hard to answer because if, say, $x = \frac{2}{3}$, then $\frac{x}{2x-1}$ becomes $\frac{\left(\frac{2}{3}\right)}{\left(\frac{1}{3}\right)}$, a 'compound fraction', which would seem *not* to be a fraction, according to the 'integer' definition of a fraction, because $\frac{2}{3}$ and $\frac{1}{3}$ are not integers. However, it *simplifies* to a fraction, $\frac{2}{1}$, which simplifies to an integer, 2 (is 2 a fraction?). So, it seems difficult to say whether $\frac{x}{2x-1}$ is or is not a fraction, even for some specific values of x, as it may depend on at what stage of the process of simplification you look.

It seems that this apparently simple 'integer' definition of a fraction introduces quite a lot of complexity if we want to be consistent in our use of language. And, in more advanced mathematics, 'partial fractions', 'continued fractions', and so on, are all *not* fractions by this definition, even though we call them 'fractions' as part of their name.

Defining a fraction can lead to some perhaps peculiar implications, since 'fraction' can refer to a particular visual format and representation of the number. Something can be *equal* to a fraction but not a fraction (e.g., 0.5 or $\frac{1}{1.5}$ or $1 - \frac{1}{3}$). So, we might say that a number like 2, that isn't a fraction, can be 'written as' a fraction ('fractionalised'), by writing it as $\frac{2}{1}$. Whereas 'rational number' refers to the number itself, however it is expressed, 'fraction' refers to a particular way of writing a number.

This suggests a different, much less strict definition of a fraction. What if we were to say that, instead of

$$\frac{\text{integer}}{\text{non-zero integer}},$$

a fraction is merely

$$\frac{\text{any expression}}{\text{any expression}}?$$

This way, any division written in this vertical format would count (see Hewitt, 2009, pp. 95-96), and the 'any expressions' could themselves be fractions, algebraic letters or even irrational or complex numbers, if we wish. We can think of this more inclusive definition as a kind of *typographical 'Equation Editor'* definition, based purely on the visual representation on the page.

This 'typographical' definition of a fraction means that we can consider a "fraction of" any number to be, say, $\frac{1}{\pi}$ of that number, without worrying that π is not an integer. It also means that if we wished to create equivalent fractions, by writing something like

$$\frac{4}{6} \times \frac{2.5}{2.5} = \frac{10}{15},$$

we could regard this as simply an example of 'fraction × fraction = fraction'. It would be exactly analogous to

$$\frac{4}{6} \times \frac{3}{3} = \frac{12}{18}.$$

With the 'integer' definition of fraction, we would have to think of $\frac{2.5}{2.5}$ as 'a division but not a fraction' and to make a distinction between these two, similar 'cancelling up' situations.

The 'typographical definition' also has the advantage that if the teacher asks a question like "Give me some fractions equivalent to $\frac{2}{3}$", then they can accept a much wider range of responses, including things like $\frac{1}{1.5}$. On the other hand, it does have the disadvantage that tasks like the one below work less well:

How many solutions can you find for $\frac{2}{\square} = \frac{\square}{24}$?

How do you know that you have them all?

Now change the '2' and the '24' so that there are:
(i) more than 20 solutions (ii) exactly 20 solutions.

With the 'integer' definition (even if we don't assume that the integers have to be positive), we get a *finite* number of solutions and a task like this works, which it doesn't with the 'typographical' definition.

There are some parallels in this discussion with the question of whether something like $\frac{2}{3}$ should be considered to be a 'factor' of 8. In a sense, it is, because $8 \div \frac{2}{3} = 12$, which is an integer. But 'factors' are often defined on the positive integers, in which case $\frac{2}{3}$ would be ineligible. However, there are cases where it is natural to be more inclusive in our meaning of 'factor'. For a statement like 'multiple congruent triangles will fit together at a point if one of the angles in degrees is a factor 360', really 360° here is an arbitrary choice for representing a full turn, and a triangle containing an angle of $\frac{2}{3}^\circ$, for instance, would also be absolutely fine. Negative factors are often invoked when factorising a quadratic expression like $x^2 - 9x + 8$, where it may be helpful to think of one of the factor pairs of 8 as being –1, –8, and this also comes into play with the *factor theorem*.

In algebra, we might use the word 'factor' to refer to, say, x as a factor of an expression like $x^3 + x$, enabling us to write it as $x(x^2 + 1)$, but in doing this we would not be implying that x was necessarily an integer, or even rational. Similarly, with a product such as $10 = 2.5 \times 4$, it could be strange to say that the 4 is a factor of 10 but the 2.5 is not. To be consistent across these cases, maybe we should qualify the word 'factor' by saying 'integer factor' or 'real factor', as appropriate? And maybe we could do something similar for 'fraction', and speak of 'integer fractions' or 'real fractions' (compare with 'decimal fractions', 'improper fractions' or 'algebraic fractions')?

In conclusion, perhaps some things, even in mathematics, are inherently fuzzy, and sometimes it makes more sense to live with that than to try to insist on a precise definition. Precise definitions seem to be much more important for some mathematical concepts than for others. Just as in everyday life, the most familiar objects, like a cat, may be the hardest to define precisely – and sometimes, even in mathematics, it may not be worth the trouble. So, our agenda here has been not really so much to try to conclude what the 'best' definition of a fraction might be, but rather to problematise the idea that mathematics, when taught 'properly', involves always having precise, consistent definitions for everything. Perhaps this is unachievable, not just for highly-advanced technical concepts, but even with something as apparently basic as a 'fraction'. More broadly, it seems important to decide which are the things we really need to fuss over, and be super-precise and careful about, and which are the things we can fudge, because they don't really matter.

Notes

1. Sometimes a trapezium is defined as a quadrilateral with *exactly* one pair of parallel lines, and sometimes it is defined as a quadrilateral with *at least* one pair of parallel lines (so any parallelogram would be an example of a trapezium) (see Foster, 2014).
2. This definition means that the rationals include the integers (e.g., $5 = \frac{5}{1}$) and recurring / terminating decimals, which can be written as a fraction (e.g., $2.\dot{1} = \frac{19}{9}$), but not *non*-recurring, *non*-terminating decimals like $\sqrt{3} + 1$ and 2π.
3. We might think usefully in terms of *global* versus *local* definitions. For example, we might say: "Today, I'm thinking about fractions where the numerators and denominators are integers". But is this saying that this is what a fraction *is* (at least for today) or just that we are today dealing with a certain subset of all fractions, which we might call 'integer fractions'?

References

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Dear Editors,

Comment on Geometry Problem 9

I missed looking at this when it first appeared. As suggested, the solution given in *Mathematics in School* 51, 3 does seem less than elegant. I offer the alternative below, although it uses two theorems which are probably less well-known nowadays!

With the notation of the diagram given:

- The circle on BE as diameter passes through A, since $\angle BAE$ is a right angle.
- Writing the side of the square as $4a$, and EA as b, from $DM^2 = DE.DA$, we have $4a^2 = 4a(4a - b)$, and hence $b = 3a$ and $2R = 5a$ (Pythagoras' theorem).
- Now consider the isosceles triangle BEF, where F is the reflection of E in AB. Its in-circle is the circle of radius r, its area is $EA.AB = 12a^2$, and its semiperimeter is $EA + EB = 8a$, so $r = 3a/2$, and $r/R = 3/5$.

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