2 *Checking for Understanding*

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In this chapter, I argue that 'checking for understanding' is impossible; all we can ever do is check for *mis*understandings. By using examples relating to Pythagoras' Theorem, I argue that *correct* answers are open to many interpretations, but *incorrect* answers (so long as the questions are well designed) can give really precise, valuable information about students' *mis*understandings. We can never conclude that a student has perfect understanding of something, but we can devise ever-trickier tasks that will find the weaknesses in their understanding, and, once we have found them, that's when the learning can begin.

Checking for understanding is impossible; all we can ever do is check for *mis*understandings. What do I mean by this?

Suppose we are interested in students' understandings of something like Pythagoras' Theorem. We could ask them to find the length of the missing side in Figure 1 [Note 1].

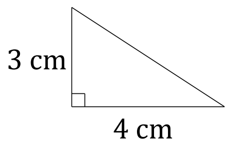

Figure 1: A 3-4-what triangle

Suppose they get this right, and say 5 cm. What does this tell us? I think it tells us very little. Maybe they know "a 3-4-5 triangle" as a thing, and so are just recalling this as a fact. Or maybe they are just mindlessly continuing the pattern 3, 4, … and saying 5. Or maybe it was a lucky guess. Were they even attending to the fact that this is a *right-angled* triangle or that Pythagoras' Theorem was relevant? Have they even heard of Pythagoras' Theorem? Even if you ask the student to show their working or explain their answer (and even if they do so!), how do you know that they are not merely reproducing something that they have remembered, with little understanding involved. Getting a question right often tells us very little about a student's understanding. Getting a question *wrong*, on the other hand, can be much more informative about *mis*understandings, and I think this is what is really useful educationally.

Suppose that the student gets that question right. We could go on to ask them to find the missing length in Figure 2.

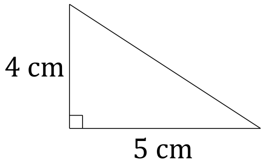

Figure 2: A 4-5-what triangle

If they answered "6 cm", that would give us a pretty clear idea that their misunderstanding was to do with simplistic pattern-following, based on the 3-4-5 from Figure 1. On the other hand, if they answered correctly, by calculating $\sqrt{4^2 + 5^2} = \sqrt{41}$ cm, I think this wouldn't tell us very much about their understanding of Pythagoras' Theorem. We can't tell from this answer how formulaic their knowledge might be, or how inert or inflexible it could prove itself under different circumstances. We don't learn much about their understanding until they get something *wrong*.

Let's suppose we go on to ask the student to find the missing length in something like Figure 3, where the required side is a leg, rather than the hypotenuse. If they calculated $\sqrt{5^2 + 6^2} = \sqrt{61}$ cm, then we could see exactly what was going wrong, as they would be failing to distinguish Figure 3 that they were given from Figure 4 that they weren't.

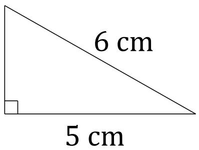

Figure 3: A 5-6-what triangle

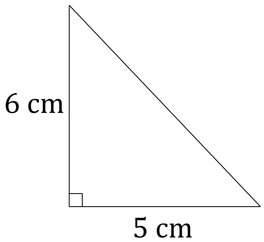

Figure 4: Another 5-6-what triangle

Alternatively, we might ask them whether they can find the missing length in something like Figure 5.

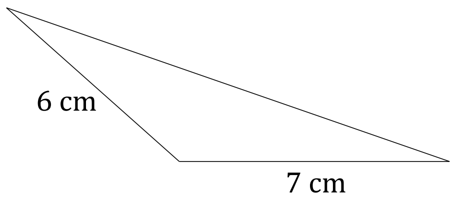

Figure 5: A 6-7-what triangle

This time, there is no right angle, so all that can be said is that, from the triangle inequality, the length of the third side must be somewhere between 7 – 6 = 1 cm and 6 + 7 = 13 cm, and without more information it is not possible to calculate an exact length. But, if the student were instead to answer $\sqrt{6^2 + 7^2} = \sqrt{85}$ cm, then we would learn something useful about the student: that they were not attending to the presence or absence of right angles in the triangles or to the uniqueness of the specified triangle [Note 2].

There is no limit to this ongoing task of posing increasingly demanding questions, stress-testing students' understanding until we find the cracks where it fails. If they get a task right, we resist jumping to conclusions about their understanding, and instead we pose something more challenging; if they get it wrong, we get valuable information that we can use to help them learn something.

Here are some tasks that, in different ways, surface different aspects of Pythagoras' Theorem, in some cases concealed within increasingly elaborate disguises:

1. Find the distance between (2, 10) and (5, 14).
2. Find the distance between (2, 10, 5) and (4, 13, 11).
3. Draw a line segment from (2, 10) to (5, 14).
   Add three more line segments to make a square.
   Find the area of the square.
4. The diagram in Figure 6 shows two concentric circles and a line segment of length 3 which is a tangent to the smaller circle.
   Find the area of the shaded annulus.

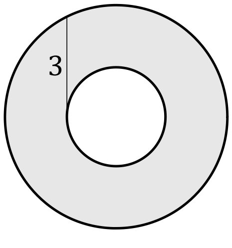

Figure 6: Two concentric circles

5. A ladder of length 13 feet is standing upright against a wall. If the top end of the ladder slides down the wall 1 foot, how far out from the wall will the bottom end move?
6. A cable 1 km long is lying flat along the ground, with its ends fixed. If its length is increased by 1 m, but the ends are still fixed 1 km apart, how high up can the midpoint of the cable be raised before the cable becomes taut?
7. I am standing in a rectangular hall, and my distances from three of the corners are 6 m, 9 m and 10 m. How far am I from the fourth corner? [Note 3]
8. What is the shortest distance from one corner of a 3 × 5 × 6 cuboid to the opposite corner, *travelling only along the surface of the cuboid?* (See Foster, 2019)

Successfully answering any or all of these questions does not, I think, in any sense 'prove' understanding of Pythagoras' Theorem. I think we should never conclude that a student 'understands', because there is always another question that might be asked, some change that might be made that *would* trip them up. You never get to the point where you are justified in placing a tick in a spreadsheet against "Understands Pythagoras' Theorem". No one ever has total understanding of anything in every conceivable situation. But this is OK. What matters in educational terms is that *failure* with any of these questions *does* reveal students' limitations, and provides opportunities to deepen the understanding that exists.

The idea that a student getting a question correct may mean very little, whereas wrong answers can be highly informative, is for me reminiscent of Karl Popper's philosophy of *falsificationism* (see Dienes, 2008, for a nice summary). You are constantly trying to falsify your belief that the student understands something by throwing increasingly tricky questions at them. If the question succeeds in tripping them up, you have succeeded in discovering a weakness in their understanding. We all have weak spots in our understanding, and all it takes is a well-designed question to surface those weaknesses and make them visible – so that we can do something about them. If the question fails to trip up the student, you don't conclude that the student 'has understanding'; you throw them a trickier question. When a student gets something right, we are of course pleased, but we don't draw sweeping conclusions from it; when they are wrong, provided the question was well designed, we have positive evidence of a difficulty that we can then address. This means that the job of learning is never finished; there is always another question to be posed. But, viewed this way, testing is not the enemy of learning. Constantly testing students by offering them challenging questions and tasks is precisely how we uncover difficulties that we can then subsequently address.

It may seem unkind to try to trip up your students, but actually it isn't. An analogy I sometimes use relates to an occasion when I was fitting shelves to the wall in my study. I carefully screwed up the brackets, and then, before loading the shelves with all my books, I pulled as hard as I could on the brackets to see if I could pull them down. I had just put the brackets up, so, in a sense, of course I didn't want them to come down. But I tried *really hard* to pull them down. If they were going to come down, I would rather that they did so now than after I had carefully arranged all my books on them! So, I was not pulling just a little bit, so as to be gentle with my precious handiwork: I was pulling as hard as I could. For me, this speaks to the idea that robust testing is the kindest thing to do. Failing to test those brackets at that point is setting up the shelves for far more disastrous failure later. Likewise, failing to test students properly when we teach them something is not being kind; it is almost guaranteeing that later on (whether this is when they are at home, trying to do their homework unaided, or in a high-stakes examination situation) they are going to come unstuck – and there will be no one there to help them then. Far better to come unstuck in the safety of the classroom, where difficulties can be addressed in a supportive environment. So, we don't give students easy challenges that we are confident they will succeed on; we test hard to find the weaknesses, so that we can help them.

In his closing address at the 1997 Association of Teachers of Mathematics Conference, Phil Boorman (1997, p. 40) talked about reconceiving his role as a teacher away from making things easy: "My job was to find a suitable field for [the students] to explore and to sit up on the hillside above and roll down rocks for the kids to jump or climb or scramble over." Rather than smoothing the path in front of them (Wigley, 1992), the teacher's job is to make life *difficult* for the students, throwing them challenges that are well-judged and will test and hone their skills. It is a battle between student and question. When the student wins over the question, that can be motivating for them, and of course we want to celebrate that, but when the question wins over the student, that is when the learning opportunities materialise.

Of course, this doesn't apply to high-stakes examinations. In those situations, we obviously hope that students will get the questions right, but a high-stakes examination is not a learning situation. In a learning situation, we shouldn't be asking a question or posing a task hoping, wishing and praying that they'll get it right, hinting towards the right answer with little verbal and non-verbal nudges. A right answer tells us very little – maybe just that our question was too easy? Right answers are great mathematically – I disagree with telling students that getting the right answer doesn't matter – but *wrong* answers are the useful ones pedagogically. If the teacher sees wrong answers as annoying but inevitable interruptions to the flow of the lesson, that use up precious classroom time, their priority will be to try to minimise the disturbance and avoid the lesson being derailed, so they can get back on track as quickly as possible. But, if we see our questions as deliberately seeking to catch the students – yes, 'trick questions', even – then we will be delighted when one of these questions succeeds, and our conjecture that students might struggle with it is borne out.

Diagnosing a disease is a positive thing when a treatment is available, and should not be something to fear. If you wish to learn, then you want to have your difficulties exposed, so you can enjoy thinking about them and become more competent – this is the culture we need to cultivate: *"Getting something wrong is great – it means you're about to learn something!*" I have heard teachers being given the advice to "Try not to show your disappointment when a student gets something wrong". Even better than this, I would say, *"Don't actually be disappointed*!"

Notes

1. The figures in this article are deliberately not drawn to scale.
2. Of course, although the figures are not drawn to scale, the student might interpret Figure 5 as clearly indicating an obtuse-angled triangle, and therefore might conclude that the missing side must be *greater than* $\sqrt{6^2 + 7^2} = \sqrt{85}$ cm.
3. See Foster (2003) for solutions to Questions 5-7.

Acknowledgement

Parts of this chapter are based on a plenary I gave at the Mathematical Association Annual Conference in April 2021.

References

Boorman, P. (1997). 'Believing is seeing', *Mathematics Teaching*, 160, 38-43.

Dienes, Z. (2008). *Understanding psychology as a science: An introduction to scientific and statistical inference*. Palgrave Macmillan.

Foster, C. (2003). *Instant Maths Ideas for Key Stage 3 Teachers: Shape and space*. Cheltenham: Nelson Thornes.
https://www.foster77.co.uk/instantmathsideas.htm
https://www.foster77.co.uk/2.07%20Pythagoras'%20Theorem.pdf
Foster, C. (2019). Spider on a cuboid. *Teach Secondary*, *8*(6), 116–117. https://www.foster77.co.uk/Foster,%20Teach%20Secondary,%20Spider%20on%20a%20cuboid.pdf
Wigley, A. (1992). 'Models for teaching mathematics', *Mathematics Teaching*, 141, 4-7.