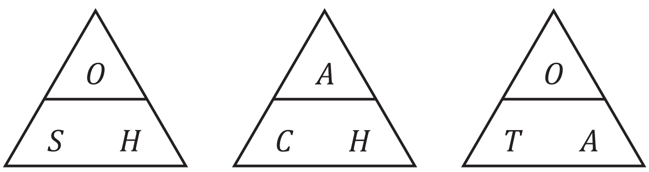
Trigonometry without SOHCAHTOA

By **Colin Foster, Tom Francome and Chris Shore**

In mathematics teaching, mnemonics are often looked on with suspicion. Teachers who use 'FOIL' for expanding a pair of brackets or 'KFC' for division of fractions are seen as resorting to 'tricks', which are merely examination-passing hacks (Cardone, 2015). However, we don't see mnemonics as necessarily always a sign of 'procedural' teaching. They give cues which may sometimes be useful, especially for arbitrary knowledge. In this article, we consider how this might apply to using 'SOHCAHTOA' for trigonometry. Should we shun the mnemonic or not?

The SOHCAHTOA way

As most readers will know, SOHCAHTOA is a commonly-used mnemonic that gives the definitions of the sine, cosine and tangent ratios in terms of the sides of a right-angled triangle: Sine is Opposite over Hypotenuse, Cosine is Adjacent over Hypotenuse and Tangent is Opposite over Adjacent (Foster, 2023). Sometimes, SOHCAHTOA is presented within formula triangles (Foster, 2021) as:



This means that students can bypass the rearranging of equations and go straight to the appropriately rearranged form that they need, such as

$$\text{hyp} = \frac{\text{adj}}{\cos 35°},$$

as the first line of their solution.

For us, the main problem with SOHCAHTOA is not that it's a mnemonic but that applying it can be complicated, and it tends to divorce the word 'sin' from 'sine as a function of an angle'. To be successful with SOHCAHTOA, students need to label their right-angled triangle correctly with 'hyp', 'opp' and 'adj', depending on which acute angle is the angle of interest, decide which is the *irrelevant* side (which is neither given nor required in the question), and then find which of the three trigonometric ratios connects the other two sides. They then need to find the appropriately rearranged formula which provides the quantity of interest, substitute in the given values (ensuring that the calculator is in degrees mode and that they haven't inadvertently evaluated something like $10\cos(35°)$ as $\cos(35°\times 10)$ or even $35\cos(10°)$), round their answer correctly, and provide the appropriate units.

There is a lot going on here, and it can be a considerable challenge to get it all right. One approach is to treat this as a long list of micro-steps to teach and rehearse in isolation, followed by chaining them together, and then practising them in larger and larger chains until the whole process is routine. This seems to us like a difficult way to proceed.

The unit-circle way

A unit-circle approach to trigonometry (Hewitt, 2007) bypasses difficulties associated with starting with definitions in the form of quotients. Instead, we define the sine and cosine as *dimensionless lengths* within a circle with unit radius. Doing this has the added benefit of permitting the argument of the function (i.e., the angle) to be *any real number*, not necessarily an angle between 0° and 90°. Our sense is that many schemes of learning have now moved towards 'unit-circle' approaches for introducing trigonometry – no longer leaving this until the sixth form – but there are still some different ways to go about this.

The basic idea is to place a circle of radius 1 with its centre on the origin, and then *define* the cosine of the angle θ in Figure 1 as the blue x-distance and the sine of the angle as the red y-distance. If you want a mnemonic for this, then alphabetically x comes before y, and cosine comes before sine. It's important that students realise that $\cos\theta$ stops at the foot of the perpendicular line and doesn't continue to the circumference of the circle, otherwise it would be equal to 1. One way to avoid this confusion is to show $\cos\theta$ extending from the end of the radial arm (at the circumference) horizontally to the

y-axis, as the blue dashed line shown below.)

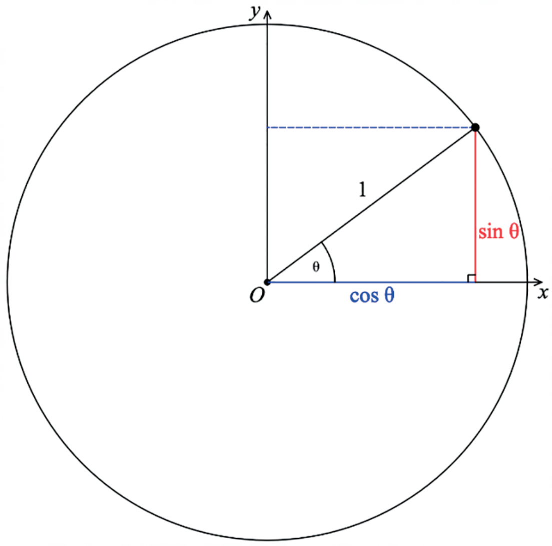


Figure 1. Defining cos θ and sin θ as lengths in a unit circle.

A possible way to begin is to create a mental image (Francome, 2012). Imagine a circle with a point on the circumference at the extreme right. Move this point slowly anticlockwise around the circle. How would you describe where the point is? When is the point highest? We can call the height here 1. When is the point lowest? We can call the height here –1. When will the height be 0? (See Hewitt, 2007, for a fuller exemplification of working with this image.)

Students can go on to watch an animation of this in *Geogebra*, in which a rotating radius vector moves from $\theta = 0°$ to $\theta = 360°$, and beyond. *Geogebra* sketches for all aspects of this approach are at **www.geogebra.org/m/d6wvwdfp**. Most mathematics animations we've seen go far too fast. It seems much more beneficial to have the animation going extremely slowly, so that there is plenty of time to talk about what's happening as it happens, and – importantly – to predict what's going to happen before the moment is spoiled by seeing it. The aim should be that learners are predicting what is coming, rather than struggling to keep up. Even with a very slow animation speed, you would probably want to press the pause button before 90°, again just after 90°, and before and after 180°, 270° and 360°: "What's happening now? What's going to happen next? Why?" The teacher can ask what angle would lead to x or y being equal to $\frac{1}{2}$, and students are very likely to predict 45°, which of course is incorrect. It is interesting to explore why this doesn't happen. Through this task, students see how cosine and sine are identical sinusoidal functions, taking values between –1 and 1, and cycling out of phase with each other.

Working the relationship the other way round – beginning with the two sinusoidal functions and *creating* the circle – can also be interesting. This can be done 'live', with people as points: two students walk up and down each of a pair of perpendicular axes, and another student tracks their positions. If the first two students walk in phase, then the third student merely walks back and forth along a diagonal line. But if the first two students walk 90° out of phase (not easy!), then the third student describes a circle (Figure 2) (Note 1).

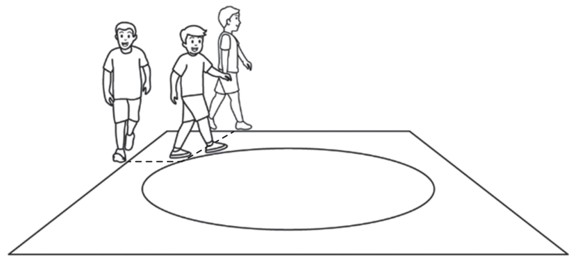


Figure 2. One student tracks the positions of the other two students as they walk back and forth along a pair of perpendicular axes.

From these visualisations, students can draw or sketch graphs of $x = \cos\theta$ and $y = \sin\theta$ and use calculators to find missing sides and angles in various different right-angled triangles (separate from their associated circles), initially all with hypotenuse 1. Eventually, if you have a set of questions in which every hypotenuse is marked as 1, someone will comment on this, or they will ask what would happen if the hypotenuse was *not* 1, but some other value. If this doesn't happen, then the teacher can pose this question. It's quite natural at this point to imagine scaling up the unit circle, so that it isn't 'unit' any more. In *Geogebra* you can drag a point on the circumference, and everything multiplies up by the same factor (otherwise it would not stay circular) – see ***www.geogebra.org/m/d6wvwdfp#material/edndknjt***. In this way, the scaling up illustrated in Figure 3 seems reasonable.

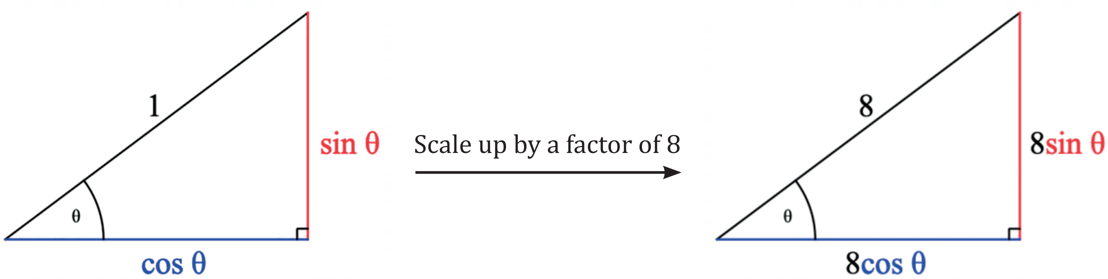


Figure 3. Scaling up a triangle with unit hypotenuse.

For us, all of this completely avoids – and is far preferable to – the SOHCAHTOA acronym. We much prefer working from the image in Figure 1, and effectively the equations adj = hyp × cos θ and opp = hyp × sin θ as the starting points, rather than the more complicated *quotients*,

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \text{ and } \sin\theta = \frac{\text{opp}}{\text{hyp}},$$

derived from SOHCAHTOA, and which usually have to be rearranged before they can be used (Note 2). All of this fits a general *linear* pattern that pervades lower secondary-school mathematics, of the form $y = mx$, where m is the multiplier (Foster, 2022). Here, the hypotenuse m determines which of the infinitely many triangles similar to the base triangle with unit hypotenuse we happen to be interested in. So the length of the hypotenuse is always the multiplier (Foster, 2021, 2022).

What about tan?

There are three options that we are aware of for incorporating tan within the unit-circle approach.

One is to just state that

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta},$$

and treat this as the starting point, building on the definitions of sin θ and cos θ already given as lengths. This means that tan θ feels quite different from the other two functions, and students may feel that tan is kind of dimensionless, whereas sine and cosine both have dimensions of length, which can be a bit of a confusion, since all three are dimensionless.

Another option is to draw on the meaning of the word 'tangent', as a line that "just touches" the curve at that point. This could be either the orange line or the purple line in Figure 4. We prefer to use the purple line, because

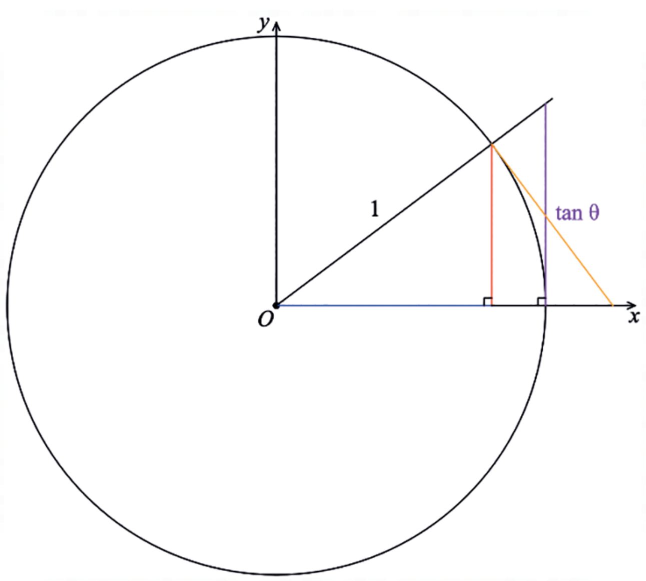


Figure 4. Defining tan θ as a length in a unit circle.

it remains vertical as the radius vector rotates, so its length is more easily compared with other relevant lengths (Note 3).

However, our preferred approach in the LUMEN Curriculum that we are currently developing at Loughborough University **www.lboro.ac.uk/services/lumen/curriculum/** is to deal with tan much earlier, *long before* introducing sine and cosine, so predating all of this formal trigonometry. Gradient is typically met in school well before trigonometry is begun, and to us the relationship $y = mx$ is the big, central unifying idea in lower secondary mathematics (Foster, 2022). When exploring straight-line graphs through the origin, $y = mx$, one point on the line (e.g., the point $(1, m)$) can be treated as controlling the position of the entire line (see **www.geogebra.org/m/d6wvwdfp#material/btyd7srp**).

When students meet the gradient, m, of a straight line, they often talk about it using 'angle language', such as the word 'angle', or they refer to the line's 'steepness' or 'slope'. So, at some point when working on lines through the origin, it can be interesting to explore the connection between the *angle* that the line makes with the x-axis and the gradient, m. At this point, we could even state that the gradient is equal to the tangent of the angle:

$$m = \tan\theta.$$

We find that even A-level students are sometimes astonished by seeing this, if they haven't encountered it before – but why not introduce it much earlier, near the start of secondary school? Tan is a nice function to explore at this point, because students always wonder about what happens when $\theta = 90°$, and $90° < \theta < 180°$. Although they wouldn't write it this formally, they also notice that $\tan(\theta + 180°) \equiv \tan\theta$, because the gradient of any line is the same when it's rotated 180° about the origin. So, there is a lot of potential for sense-making of the numbers that their calculator gives them for tan of different angles that they type in (Note 4).

Working on a grid leads naturally to *non-integer* gradients. If you have a line through the origin, and a point (a, b), where a and b are co-prime (Figure 5), then the relationship between the equation

$$y = \frac{b}{a}x$$

and the gradient,

$$\tan\theta = \frac{b}{a}$$

is transparent. This can be much more intuitive than beginning with 'nice' integer gradients (see **www.pmtheta.com/uploads/4/7/7/8/47787337/dose_of_don_11th_nov_2020.docx**).

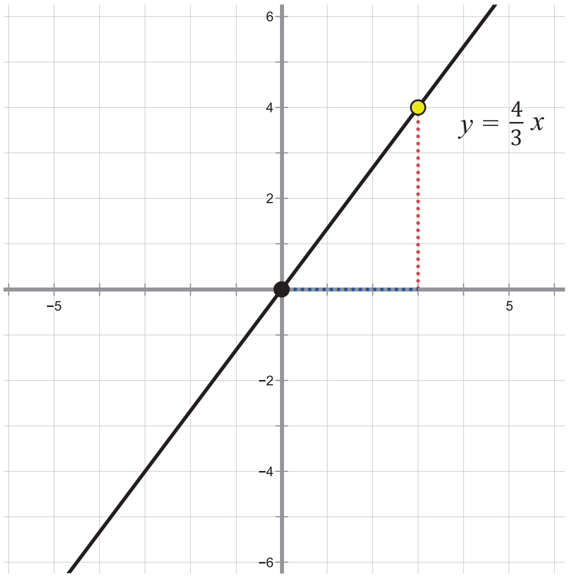


Figure 5. Controlling the line $y = \frac{b}{a}x$ by manipulating the point (a, b), where $a = 3$ and $b = 4$.

Having done all of this work previously means that when we do arrive at the unit circle and begin the more formal study of trigonometry, students have already met the idea that the gradient of any line through the origin will be tan of the angle with the horizontal. From this,

$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

follows naturally, and the purple line in Figure 4 can then be a way to see that the gradient is equal to the vertical height of a right-angled triangle with unit base. However we introduce tan, it is inevitably a much more complicated function than sine and cosine (i.e., not sinusoidal or bounded, and with vertical asymptotes), so asking students to figure out its shape from what they know about sine and cosine can be a difficult but rewarding activity.

It seems to us that the unit-circle approach offers much more than 'SOHCAHTOA' does. It is more forward-looking for students who will go on to study trigonometry further, it relies less on memorisation, it reinforces manipulations of equations, and it makes strong connections with gradient and with the graphical forms of the circular trigonometric functions.

Notes

1. It would of course be possible to explore other Lissajous curves as well – see **https://github.com/mnerv/Lissajous-Curve-table-macOS**.

2. Having these in the form 'hyp × trigonometric function', rather than 'trigonometric function × hyp', matches the general $y = mx$ pattern that we seek to stress everywhere, where the multiplier, m, always comes first. It also has the advantage that the calculator error of, for example, $\cos(\theta \times \text{hyp})$ is rendered less likely. This form, rather than quotients like $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, is also the more useful one in applied situations, such as A-Level mechanics, where you often want to resolve a vector into perpendicular components, whereas you rarely need to find a hypotenuse.

3. The orange line segment is also useful, however. For example, when teaching the cot function, the line segment for cot joins up with the orange one, making the other 'half' of the same tangent line. Similar arguments can be made for the other reciprocal functions – see **www.geogebra.org/m/d6wvwdfp#material/kxcnayst**.

4. A similar kind of approach can be used for the minor/reciprocal trigonometric functions (see **www.geogebra.org/m/d6wvwdfp#material/kxcnayst** with all the boxes checked). Seeing these helps students see our choices, such as working with $\frac{\text{opp}}{\text{hyp}}$ rather than $\frac{\text{hyp}}{\text{opp}}$, as less arbitrary.

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Keywords: Definitions, Gradient, SOHCAHTOA, Trigonometry, Unit circle, Visualisation.

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