Acknowledgements

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Note

1. After a question proposed by a student: can you show a particular result, involving a trigonometric limit, using the definition of limit? The answer to this question is the aim of this manuscript.

Reference

[1] Henle JM, Kleinberg EM. Infinitesimal calculus. New York: Dover; 1979.

Mathematical études: embedding opportunities for developing procedural fluency within rich mathematical contexts

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In a high-stakes assessment culture, it is clearly important that learners of mathematics develop the necessary fluency and confidence to perform well on the specific, narrowly defined techniques that will be tested. However, an overemphasis on the training of piecemeal mathematical skills at the expense of more independent engagement with richer, multifaceted tasks risks devaluing the subject and failing to give learners an authentic and enjoyable experience of being a mathematician. Thus, there is a pressing need for mathematical tasks which embed the practice of essential techniques within a richer, exploratory and investigative context. Such tasks can be justified to school management or to more traditional mathematics teachers as vital practice of important skills; at the same time, they give scope to progressive teachers who wish to work in more exploratory ways. This paper draws on the notion of a musical étude to develop a powerful and versatile approach in which these apparently contradictory aspects of teaching mathematics can be harmoniously combined. I illustrate the tactic in three central areas of the high-school mathematics curriculum: plotting Cartesian coordinates, solving linear equations and performing enlargements. In each case, extensive practice of important procedures takes place alongside more thoughtful and mathematically creative activity.

Keywords: Cartesian coordinates; enlargement; fluency; learning for understanding; practice; solving equations; task design; traditional and progressive approaches to the teaching of mathematics

1. Introduction

Traditional approaches to the teaching of mathematics emphasize repetition as a route to confident fluency. Failure to recall important facts, and lack of procedural efficiency with

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critical techniques, it is argued, hamper learners' subsequent mathematical development.[1] Consequently, the teacher who does not provide sufficient opportunities for consolidation of processes does their learners a grave disservice (see [2–4]). On the other hand, progressives object strongly to the stultifying nature of laborious drill-and-practice approaches to the subject, arguing that real mathematical learning must involve open-ended, non-standard problems and must call on learners to be inventive and creative, exploring the mathematical terrain in their own way.[5] To devote precious classroom time to having learners work through pages of routine exercises is a travesty of what authentic mathematical learning should be.[6] The result, according to Noyes,[7] is that 'Many children are trained to do mathematical calculations rather than being educated to think mathematically' (p.11).

Most mathematics teachers would probably put value on both of these aspects, and anecdotal evidence suggests that many thoughtful teachers seek to compromise by offering a balance of tasks of both kinds. However, in a high-stakes assessment culture, the backwash effect of examination requirements is likely to drive mathematics teachers away from investigative work and towards endless practising of 'the finished product'.[8] Exhorting mathematics teachers to offer more investigative tasks may be viewed as impractical and simply generate frustration with the system and feelings of guilt.

This paper borrows from music the notion of an *étude* in order to attempt to harmonize the apparently conflicting requirements of developing learners' procedural fluency and offering richer, more investigative mathematical experiences. I explore how features of this musical genre can inform the design of mathematical tasks, and I illustrate how the approach can be implemented in three key areas of the high-school mathematics curriculum: plotting Cartesian coordinates, solving linear equations and performing enlargements. In each case, I suggest that plentiful practice of important procedures can be embedded within much richer and more thought-provoking mathematical activity, and I discuss some of the affordances of doing this.

2. Mathematical études

According to the Encyclopaedia Britannica, an *étude* is 'originally a study or technical exercise, later a complete and musically intelligible composition exploring a particular technical problem in an esthetically satisfying manner'.[9] Early études were for private practice and not intended for performance, but later ones (sometimes termed *concert studies*) sought to achieve the twin objectives of pleasing an audience in concert as well as operating as an effective teaching tool. It is in this latter sense that I will employ the term *étude* in this paper, since my purpose is to inform approaches to designing mathematical tasks that embed extensive practice of a well-defined mathematical technique within a richer, more aesthetically pleasing mathematical context. It is the later musical études, such as those by Chopin, that are particularly well known and loved today, the self-imposed constraint of focusing on (normally) a single specific technique perhaps being partly responsible for the beauty of the compositions.

In the typical mathematics classroom, practice of techniques tends to be associated with laborious exercises which learners must endure before they are deemed ready to employ their skills in more stimulating contexts. This might be seen as analogous to the musician routinely practising their scales and arpeggios in order to develop their muscle memory for such processes. However, in mathematics, unlike in the purely physical domain, repetition of a technique that is not meaningful to the learner is unlikely to be beneficial. As Holt [10] comments:

> the notion that if a child repeats a meaningless statement or process enough times it will become meaningful is as absurd as the notion that if a parrot imitates human speech long enough it will know what it is talking about. (p.193)

Indeed, repetition by itself can *dull* learners' senses to the point that they are no longer cognitively engaged in their mathematical work, such as when learners complete pages of questions making the same error in each one. Such drill may also contribute to an alienating perspective on the value and purpose of mathematics.

On the other hand, fluency with fundamental mathematical techniques would seem to be a legitimate goal that may not be achieved by more open-ended, problem-solving approaches to the learning of the subject. It has been well argued that problem solving constitutes more authentically mathematical activity, and allows learners the space to develop independent creative solutions.[11] However, when learners have the opportunity to choose from their arsenal of different mathematical techniques those that they are going to bring to bear on a particular problem, there is always the possibility that they will avoid areas of weakness and end up succeeding in solving the problem by playing to their mathematical strengths. Thus, under-developed areas can remain so. For all their faults, tightly focused exercises do have the advantage of forcing learners to focus on specific areas for development. It is clearly important for learners to address weaknesses; Dweck [12] proposes that if a learner does something 'quickly, easily and perfectly', then instead of praising their intelligence, the teacher 'should apologize to the student for wasting his or her time with something that was not challenging enough to learn anything from' (p.121). A musician might conceal a particular technical weakness through their choice of repertoire, but an étude focusing on that one particular skill gives them nowhere to hide. Noddings [13] proposes that 'Drill should be used judiciously – to routinize skills that will make the learning of important concepts easier and more enjoyable' (p.123).

Csikszentmihalyi [14] suggests that 'It is a mistake to assume that creativity and rote learning are incompatible. Some of the most original scientists, for instance, have been known to have memorized music, poetry or historical information extensively' (p.123). So it would seem that the model of an étude, in seeking to address the twin needs of procedural fluency and more authentic problem solving, might potentially be useful in the domain of mathematical task design. Most musical études are short (sometimes less than a minute) and focus on the development of one key technical challenge. However, the best ones are also elegant and creative works of art in their own right. While no major composer made their name by just writing études, many did manage to produce pieces which function highly effectively in developing technical fluency while also being beautiful and intricate works of art. I will seek to draw on this analogy to elaborate some principles for the construction of mathematical tasks which embed sufficient practice of important techniques while at the same time provoking rich mathematical thinking. Figure 1 suggests that while routine exercises promote the development of procedural fluency, and more open-ended problem-solving tasks encourage exploration and creativity, a well-constructed mathematical étude might succeed in doing both simultaneously.

Combining these two aspects is not a completely new idea, but it is perhaps a timely one. Andrews,[15] considering the high-school topic of measuring angles, describes his desire for 'a means by which practice could be embedded within a more meaningful and mathematically coherent activity' (p.16) and goes on to outline exploratory tasks relating to angle properties and proof which incidentally involve learners in repeatedly measuring angles with a protractor. In such a setting, learners self-differentiate, giving more attention to the measuring aspect and performing it more slowly if they find it hard, and carrying

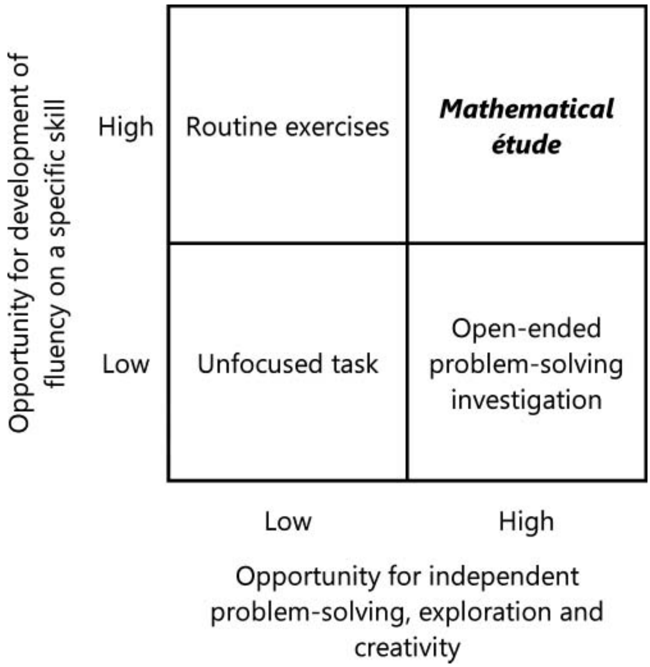

Figure 1. Different genres of mathematical task.

it out correctly and with less thought if it is a procedure that they have already mastered. Learners comfortable with the technical side can devote their spare cognitive capacity to the deeper aspects of angle properties and relationships, and so work on justification and proof. Rather than keeping some learners away from rich higher-level-thinking tasks until they have mastered prior procedures to the teacher's satisfaction, tasks such as these dignify all learners with opportunities to think mathematically.

I will now suggest four examples of mathematical études in four different areas of the mathematics curriculum and briefly consider their affordances.

3. Cartesian coordinates

The teacher who wishes their learners to rehearse conventions for using Cartesian coordinates in two dimensions might conceivably give them a random list of pairs of coordinates for them to plot. In fact, tiresome as it might sound, such exercises are by no means absent from high-school mathematics textbooks. Clearly, such a task is very likely to be experienced by the learner as a boring and purposeless activity. In an attempt to ameliorate this, these exercises sometimes result in the creation of an aesthetically pleasing picture (e.g. an animal) when the dots are finally joined with line segments. Although this may make the task slightly less irksome, the more appealing aspect of it is non-mathematical, with the feeling of having been 'bolted on'. The mathematics, such as it is, is the boring part.

To develop a task such as this into a mathematical étude it would be necessary to subsume the plotting of the coordinates within a mathematically much richer context. So, instead of providing learners with a list of random pairs of coordinates, the teacher might instead invite them to produce their own coordinates by employing a rule such as 'the second coordinate is twice the first coordinate' or 'the first coordinate is three times the second coordinate, minus two'. Such a task might now justify the designation 'mathematical étude', since practice of a particular technique is taking place, but something with far richer mathematical possibilities is happening at the same time.

I have observed in the classroom that learners are much more enthusiastic about plotting their points (and, consequently, perform the procedure many more times) when it is with a particular purpose in mind. They want to get the points in the right places, because their location matters for some wider purpose. Learners may find the patterns that result (e.g. straight lines) surprising and intriguing, and thus might be inclined to pose their own 'What if?' questions in relation to other possible rules. They will notice if a point does not fit the pattern and perhaps check whether they might have plotted the values the wrong way round or overlooked a negative sign. Such work is also more readily extendible for learners who finish early or who are particularly confident with the technique, since they can employ more complicated rules (e.g. 'the second coordinate is the square of the first coordinate'). It is more interesting for both the learners and the teacher, much less tedious for the teacher to mark, and represents a worthwhile mathematical task in its own right. It also has the potential to illustrate to learners why coordinates might be useful in mathematics and why plotting them correctly could be important.

4. Connected expressions

The design of this mathematical étude began with the aim of learners practising solving simple linear equations such as $3x + 7 = 31 - x$. How can learners get good at solving equations like this without simply being given lots of equations like this to solve? Just as with the coordinates task, one design principle that has helped me to generate such études has been to place a limit on the amount of content that I am prepared to give to learners. Just as with the traditional coordinates exercise, a traditional approach to solving equations would entail providing learners with numerous arbitrary questions on the board, in a textbook or on a worksheet. Working for a time in a school which did not have the budget for photocopying multiple sheets of questions was a helpful stimulus for me to find ways in which learners could generate their own questions from a minimal teacher starting point.

One solution is to give a framework such as

$$\square x \pm \square = \square x \pm \square$$

and to invite learners to find integers (positive or negative) to go in the boxes so that, when the equation is solved, the answer is *also* an integer.[16,p.30–33] For learners aged 11 upwards, this can be a challenging task, but it is easy for either the learner or the teacher to allow one or two boxes to be zero to start with, so as to make the task initially more accessible. The value of learners constructing their own mathematical examples to satisfy given constraints is well established,[17] and the process of trial and improvement entails plenty of practice at the key skill of solving equations, while something a bit more interesting is going on. For older learners seeking additional challenge, replacing one of the xs with x^2 would increase the difficulty considerably, and perhaps the interest level too.

A subsequent development of this idea, with similar aims, was the construction of the *connected expressions* diagram shown in Figure 2 [18]. A *connected expressions* diagram consists of n algebraic expressions with ${}^nC_2$ line segments joining every pair. Alongside each line segment, learners write the solution to the equation formed by equating those two

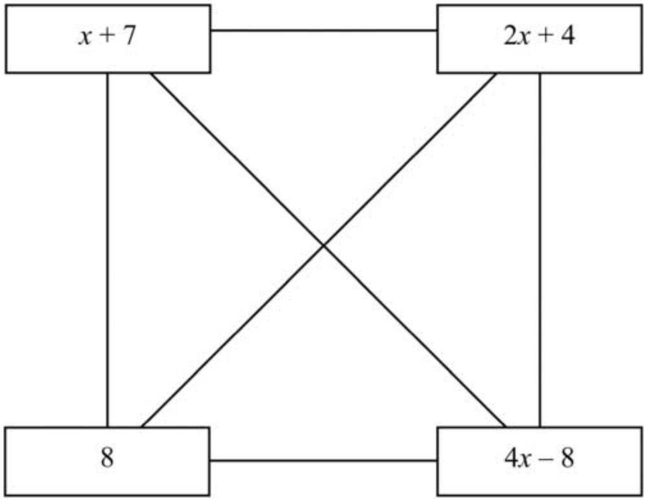

Figure 2. Connected expressions.

expressions. In the example given in Figure 2, the solutions to each of the six equations reveal the pattern $\{1, 2, 3, 4, 5, 6\}$. Offering this diagram to learners is equivalent to asking them to solve six linear equations, though with some important differences. First, less arbitrary content is given to the learners: instead of 12 expressions organized into six pairs we have just four expressions, combined in every possible pairing. Second, other areas of mathematics are available to work on, such as the number of ways of choosing two items from six and the properties of regular polygons. Third, learners have slightly more autonomy, in being able to select where to start on the diagram; i.e. which pair of expressions to begin with. There is a sense in which learners may feel that they are creating or revealing the equations rather than simply receiving them. Fourth, the pattern in the solutions obtained constitutes another level of attention. Drawing learners' awareness away from the nitty-gritty of the solving of each equation onto a slightly higher level encourages learners to develop their fluency with equations, as they are being stimulated at the same time to think about how the different expressions relate to one another and how they might have been chosen. Likewise, learners for whom solving these equations is not too demanding have a bigger problem to think about: how did the teacher choose these four expressions to make this happen? Fifth, the pattern in the answers may provoke some self-checking where one solution does not fit with the rest.

This *convergent* task, in which all learners ultimately (aside from mistakes) obtain the same six numbers, is then followed by a more *divergent* task, in which learners are invited to create their own *connected expressions* diagram. They might try to obtain a different set of six solutions, such as the first six primes or the first six multiples of 5. Initially, the challenge of obtaining all answers as integers may be sufficient, and learners may prefer to begin with a triangle (three expressions and three equations) rather than the square. Some learner creations, produced by learners aged 12–15, are shown in Figure 3. Several learners expressed surprise that, for instance, doubling all of the expressions in Figure 2 did not change the solutions obtained. Throughout both the convergent and the divergent phases of this task, learners are repeatedly forming and solving equations with a particular goal in mind, obtaining important practice at the technique while engaged in a richer investigative problem.

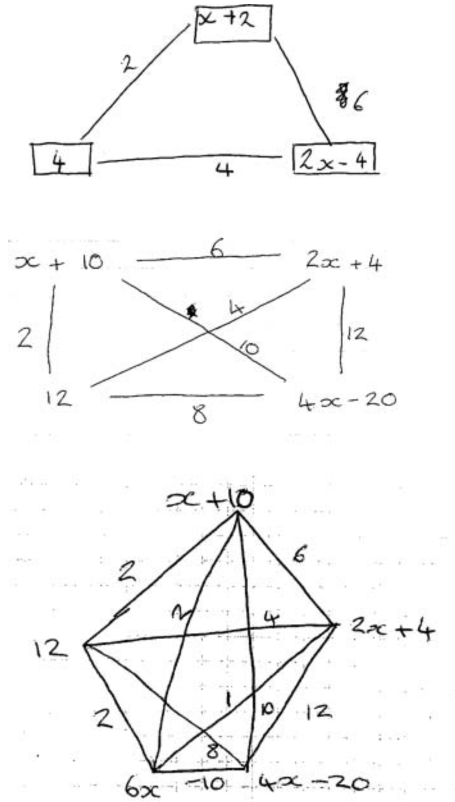

Figure 3. Learners' connected expressions.

5. Enlargements

My third example of a mathematical étude is a task in which learners are asked to enlarge a given triangle with a scale factor of enlargement of 3 about a centre of enlargement of their choice.[19] Having done this a few times, they may discover that sometimes the enlarged triangle goes off the edge of the paper. So a natural question for learners to ask, or one that can be posed to them, is 'Where can the centre of enlargement be so that the image lies completely on the grid?' Learners may conjecture that the locus of centres of enlargement such that the image *just* lies on the grid will be a triangle mathematically similar to the given one, or perhaps a circle, oval or (rounded) rectangle. As can be seen in the learner's work shown in Figure 4, much practice of drawing enlargements ensues as learners seek to establish the permissible area in which the centre of enlargement may lie. At the same time, their attention is being drawn to the edges of the paper, and perhaps to working backwards from where they want the image to lie in order to construct a possible position for the centre of enlargement. Such analysis may help them to appreciate more deeply the details of how the enlargement method works.

There is scope for confident learners to extend the problem by trying a different starting shape, or placing it in a different position on the grid, and exploring the effect that this has on the boundary of possible positions for the centre of enlargement. In this way, the étude is mathematically quite open.

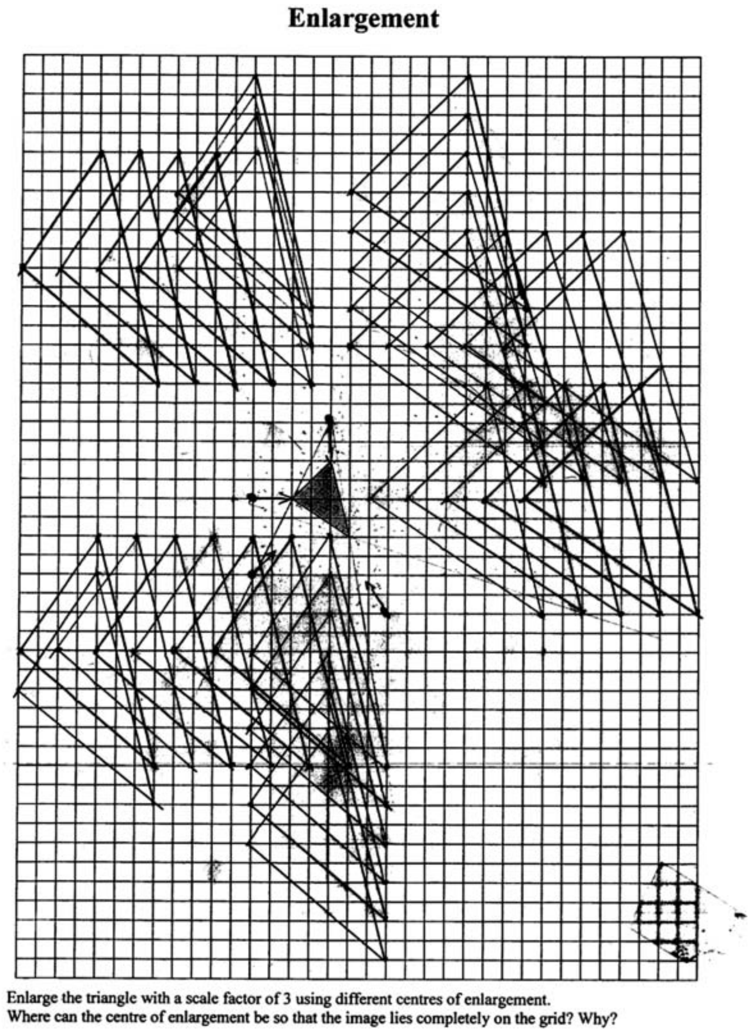

Figure 4. A learner's enlargements.

6. Conclusion

I have intended to show in these examples that the twin aims of practice for mathematical fluency and exploratory investigation of mathematical contexts are not mutually exclusive. On the contrary, it might be argued that combining them in a mathematical étude entails some additional and valuable affordances not possessed by either mathematical exercises or standard problem-solving tasks (Figure 1):

- fluency may be encouraged by deliberately taking attention away from the minutiae of the procedure and focusing on a bigger problem;
- learners who are confident with the technique (or become so during the task) have a more interesting bigger problem to focus on;

- learners' experience of mathematics is likely to be as something more purposeful and more authentically mathematical, as they see some reason why the particular technique might be useful and important.

Instead of attempting to strive over a series of lessons for a delicate mixture of procedural and exploratory activities, mathematics teachers can utilize mathematical études to 'kill two birds with one stone'. Holt [10] comments that:

> When we give children long lists of arithmetic problems to do in school, hoping to create confidence, security, certainty, we usually do quite the opposite, create boredom, anxiety, less and less sharpness of attention, and so, more and more mistakes, and so in turn, more and more fear of making mistakes. (p.75)

Practice as an end in itself can undoubtedly become stultifying and counterproductive, yet procedural fluency can in certain circumstances be an extremely satisfying result. Sacks [20] describes in glowing, even euphoric, terms the 'heavenly ease and sureness' of physical fluency following an injury: 'I knew what to do, I knew what came next, I was carried ahead by the ongoing musical stream, without any conscious thought or calculation ... The joy of sheer doing – its beauty, its simplicity – was a revelation'. He describes this experience as being 'so different, so absolutely different, from the elaborate and exhausting computation before', commenting that 'everything felt right ... with an integral sense of ease – and delight' (p.121). To have learners of mathematics experience the joy of fluent familiarity with important mathematical techniques, without sacrificing mathematical meaning, must be the aim of every teacher of mathematics. Ball and Ball [21] have pointedly asked what an equivalent event to an end-of-term school concert might be for mathematics teaching. Perhaps mathematical études, with their combination of practice and performance, might have a part to play in answering this question.

References

[1] Mathematically Correct. Toward a cease-fire in the math wars. 1998. [accessed on 21 October 2012]. Available from: http://mathematicallycorrect.com/cease.htm.
[2] Klein D. A quarter century of US 'math wars' and political partisanship. BSHM Bull. 2007;22:22–33.
[3] Schoenfeld AH. The math wars. Educ policy. 2004;18:253–286.
[4] Wright P. The math wars: tensions in the development of school mathematics curricula. Learn Math. 2012;32:7–13.
[5] Ollerton M, Sykes P. Getting the buggers to add up. 3rd ed. London: Continuum; 2012.
[6] Boaler J. The elephant in the classroom: helping children learn and love maths. London: Souvenir Press; 2009.
[7] Noyes A. Rethinking school mathematics. London: Sage; 2007.
[8] Prestage S, Perks P. Doing maths or practising the finished product. In Hewitt D, editor. Proc Br Soc Res Learn Math. 2006;26:65–70.
[9] Britannica. étude. 2012 [accessed on 21 October 2012]. Available from: http://www.britannica.com/EBchecked/topic/194671/etude.
[10] Holt J. How children fail. London: Penguin; 1990.
[11] Pólya G. How to solve it: a new aspect of mathematical method. London: Penguin; 1990.
[12] Dweck CS. Self-theories: their role in motivation, personality, and development. Philadelphia (PA): Taylor & Francis; 2000.
[13] Noddings N. Happiness and education. Cambridge: Cambridge University Press; 2003.
[14] Csikszentmihalyi M. Flow: the classic work on how to achieve happiness. London: Rider; 2002.
[15] Andrews P. Angle measurement: an opportunity for equity. Math Sch. 2002;31:16–18.
[16] Foster C. Resources for teaching mathematics 11–14. London: Continuum; 2011.

[17] Watson A, Mason J. Mathematics as a constructive activity: learners generating examples. Mahwah: Erlbaum; 2005.
[18] Foster C. Connected expressions. Math Sch. 2012;41:32–33.
[19] Foster C. Working without a safety net. Aust Math Teach. 2012;68:25–29.
[20] Sacks O. A leg to stand on. London: Picador; 2012.
[21] Ball D, Ball B. Maths is more than you bargained for. Opening address to the 2003 ATM Easter Conference, reproduced as a supplement to Mathematics Teaching 183, June 2004. [accessed on 21 October 2012]. Available from: www.atm.org.uk/journal/archive/mt183files/ATM-MT183-19-21-Extra-Ball.pdf.

Polar and singular value decomposition of 3 × 3 magic squares

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In this note, we find polar as well as singular value decompositions of a 3×3 magic square, i.e. a 3×3 matrix $\mathbf{M}$ with real elements where each row, column and diagonal adds up to the magic sum s of the magic square.

Keywords: magic squares; polar decomposition; singular value decomposition; Luoshu

1. Introduction

Any 3×3 magic square can be represented as a 3×3 matrix $\mathbf{M}$ with real elements where each row, column and diagonal adds up to s, called its magic sum, i.e.

$$\mathbf{M1} = s\mathbf{1}\,, \quad \mathbf{1}'\mathbf{M} = s\mathbf{1}'\,, \quad \mathrm{tr}(\mathbf{M}) = s\,, \quad \mathrm{tr}(\mathbf{FM}) = s\,,$$

where $'$ denotes transposition, $\mathbf{1} = (1, 1, 1)'$, and left multiplication by the (flip) matrix $\mathbf{F} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ reverses the rows of $\mathbf{M}$ such that $\mathrm{tr}(\mathbf{FM})$ is equal to the sum of the elements of the anti-diagonal of $\mathbf{M}$.

According to Trenkler, Schmidt and Trenkler,[01] such a matrix can be written as

$$\mathbf{M} = s\mathbf{J} + \mathbf{N}\,,$$

where $\mathbf{J} = \frac{1}{3}\mathbf{11}'$ is an idempotent matrix, and

$$\mathbf{N} = \mathbf{N}(\alpha, \beta) = \begin{pmatrix} \alpha + \beta & -2\alpha & \alpha - \beta \\ -2\beta & 0 & 2\beta \\ -\alpha + \beta & 2\alpha & -\alpha - \beta \end{pmatrix}$$

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