Straight to the Point

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It is always nice when an idea for a task comes from a member of the class (Silver, 1994; Kilpatrick, 1987). I had asked Year 7 students to draw several graphs of the form $y=mx+c$, where m and c are constants, choosing for themselves what values to use for the constants. The idea was to look for the effect of m and c on the shape of the graph. Making the link between the form of an equation and the appearance of its graph is something that is reported as being difficult (Knuth, 2000a, 2000b).

Some students varied their m and c values wildly, but most chose small integers and selected them reasonably systematically. One student, Kyle (a pseudonym), began with the following equations: (A) $y=3x-2$ and (B) $y=2x+1$. He noticed that the lines intersected at an integer lattice point, (3;7) (Figure 1). He began to seek another line that would pass through the same point. He explains in Figure 2 how he did this.

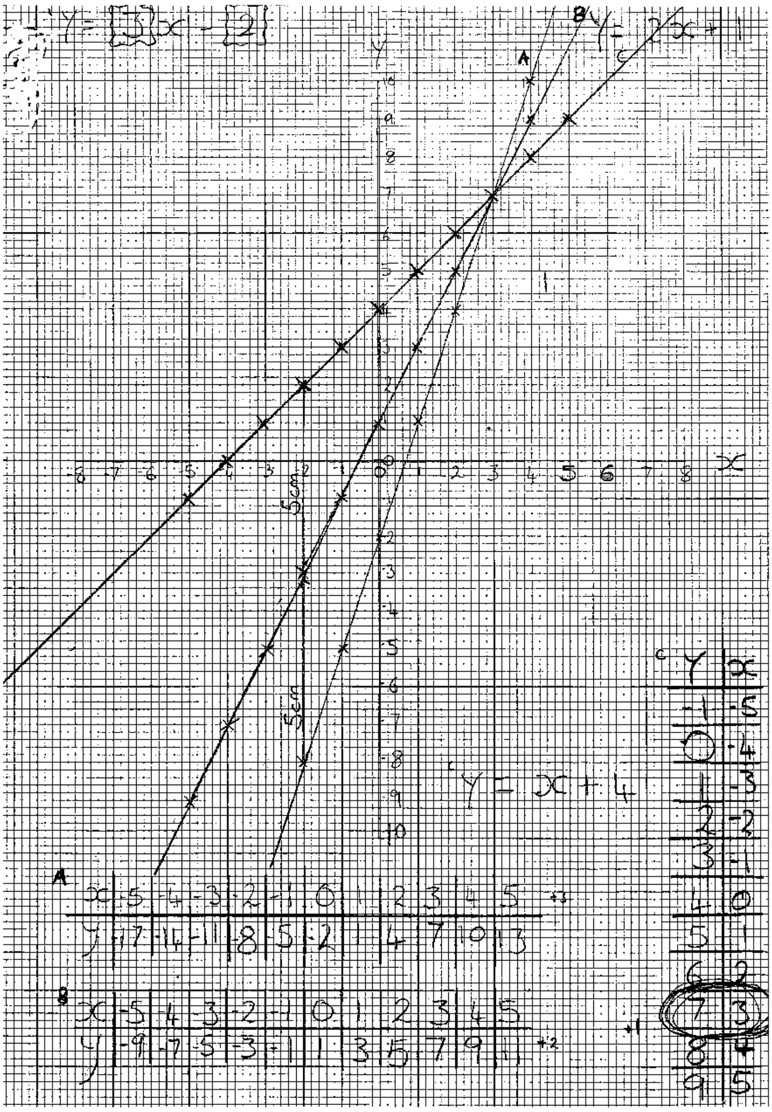
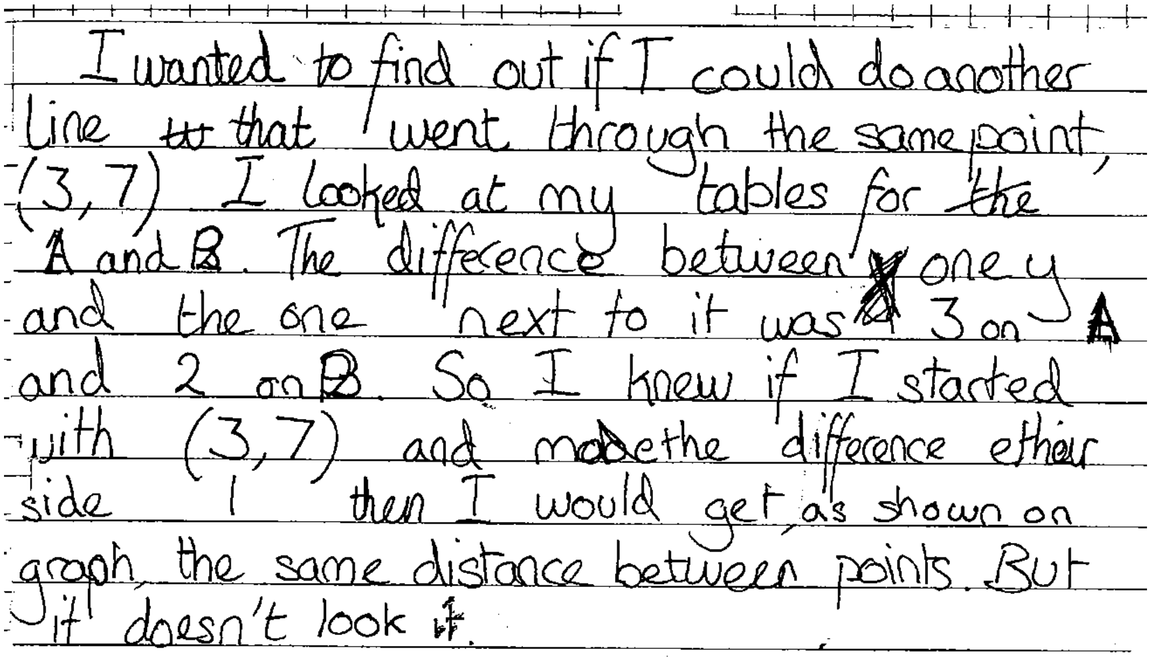


FIGURE 1



I wanted to find out if I could do another line ~~tt~~ that went through the same point, (3,7) I looked at my tables for ~~the~~ A and B. The difference between ~~y~~ one y and the one next to it was ~~A~~ 3 on A and 2 on B. So I knew if I started with (3,7) and made the difference ethair side then I would get, as shown on graph, the same distance between points. But it doesn't look it.

FIGURE 2

His third graph (C) was $y = x + 4$. He was particularly focused on the distance along each line between adjacent points. His approach to describing the slope of his graphs was different from the usual way of comparing the vertical increase for unit horizontal increase (Stump, 2001). Instead, he focused on the distance *along the line* between points plotted with unit horizontal increase. The spacing of the crosses along his line gave him a measure of slope - the closer together, the shallower the line; the further apart, the steeper the line. (This dynamic approach of running along the line and noticing how frequently the crosses appear perhaps relates to work done in science with ticker-tape, where the machine punches a hole at regular time intervals as the tape is pulled through at varying speeds. Note also that Kyle's measure of slope corresponds to using the trigonometric function $\sec\theta$ rather than the more usual $\tan\theta$.)

When he mentions "the same distance between points", Kyle explained to me that he meant the *vertical* distance between points with the same x-value on different lines. In Figure 1 he has marked '5 cm' vertically twice: once between graphs *A* and *B* and once between graphs *B* and *C*. Although these are equal, it is easy to sympathise with his comment, "But it doesn't look it"! There is an optical illusion here, where the eye is drawn to the *angle* whereas what Kyle is measuring is the vertical *distance*. This could be seen as relating to the fact that the trigonometrical function $\tan\theta$ is not a linear function, and this was an interesting outcome from this task that I had not anticipated.

Meanwhile, other students had been drawing various lines and coming to conclusions about the roles of m and c in determining the orientation and position of the line $y = mx + c$. I thought it would be interesting to exploit Kyle's interest in lines that pass through a common point to stimulate some consolidation work. I took the class to a computer room and presented them with a drawing I had created showing 12 lines all passing through the point (2;3) (Figure 3). I asked them to use graph-drawing software to recreate my picture. This turned out to be an effective way of assessing students' understanding of the impact of m and c on the position and slope of the line. I encouraged those who were having to do a lot of trial and error to refer to the previous task and attend to how the values of m and c were changing the lines.

One student describes how she tackled this task in Figure 4. (Although she uses the word 'equation' to describe just the right-hand side - where her focus was - she had to enter the '$y =$' part as well in order for the computer to draw the line.) It is interesting that she describes the finished picture as a "full circle", even though (as she must, in some sense, know) it is composed entirely of straight lines. Such an effect is reminiscent of what happens in curve stitching (Millington, 1989).

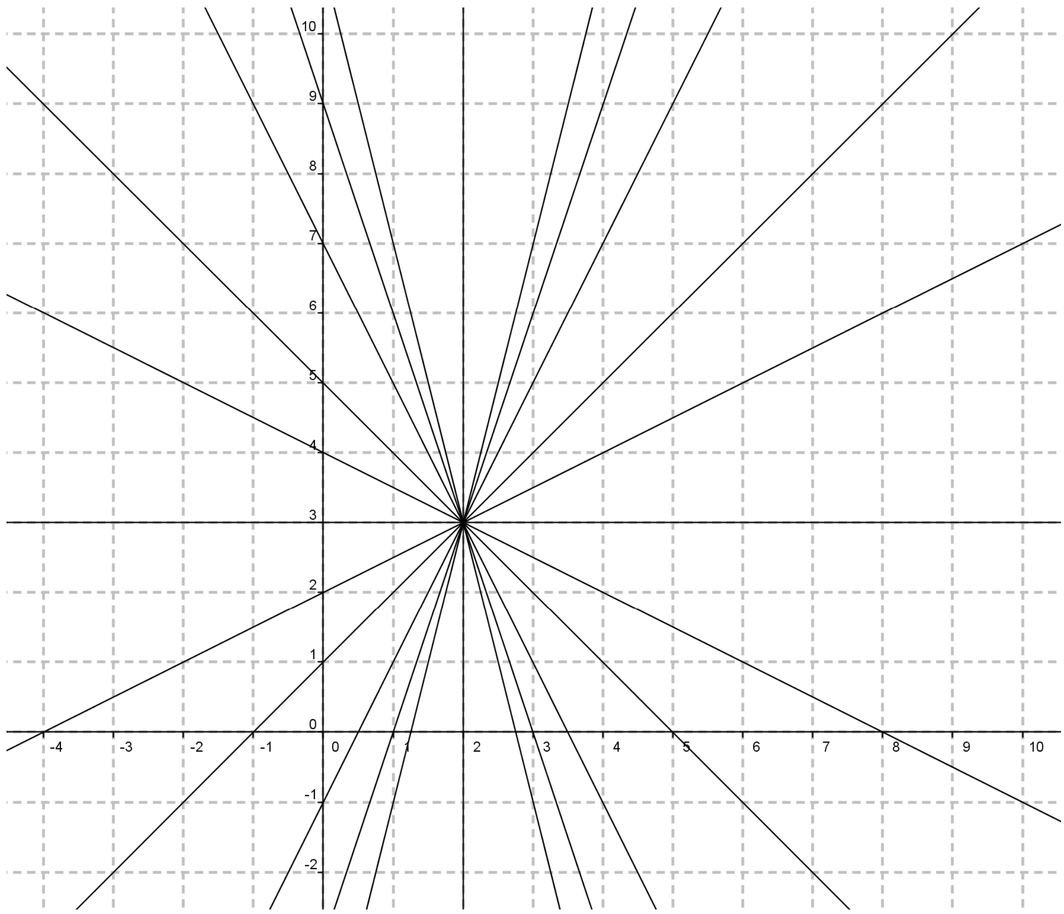


FIGURE 3

We used a graph creating programme on the Computer. We were given a piece of paper with lots of lines all coming from or going through the point (2,3).
On the computer I typed in 1x+1.
A line was drawn and it went through the point (2,3). Next I experimented until the equation 2x+-1 drew another line that went through (2,3). Then, 3x+-3 went through (2,3). Then I noticed the pattern.
You go down the x numbers in the normal number pattern (eg. 1, 2, 3, 4, 5, 6 etc.) You then put a plus sign (+) and put a minus number
The minus numbers go down in ODD numbers.
So, example,
y = 3x + - 3
y = 4x + - 5
y = 5x + - 7
y = 6x + - 9
etc.

You keep going like this until you make the full circle.

FIGURE 4

It was quick and easy to assess students' work at a glance as I circulated around the room (Figure 5). All students noticed the patterns in the numbers in their equations and several spontaneously continued beyond the 12 lines given to generate more lines passing through (2;3) (e.g., Figure 6: notice again that this student describes his picture as a 'cool curve', although it consists only of straight lines.)

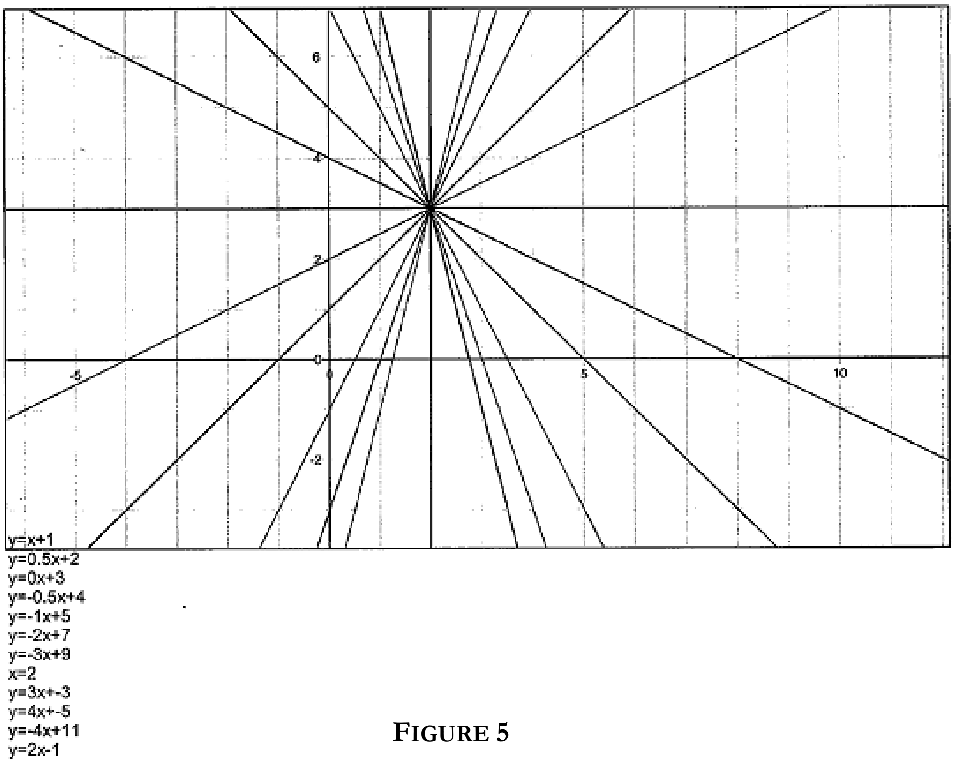


FIGURE 5

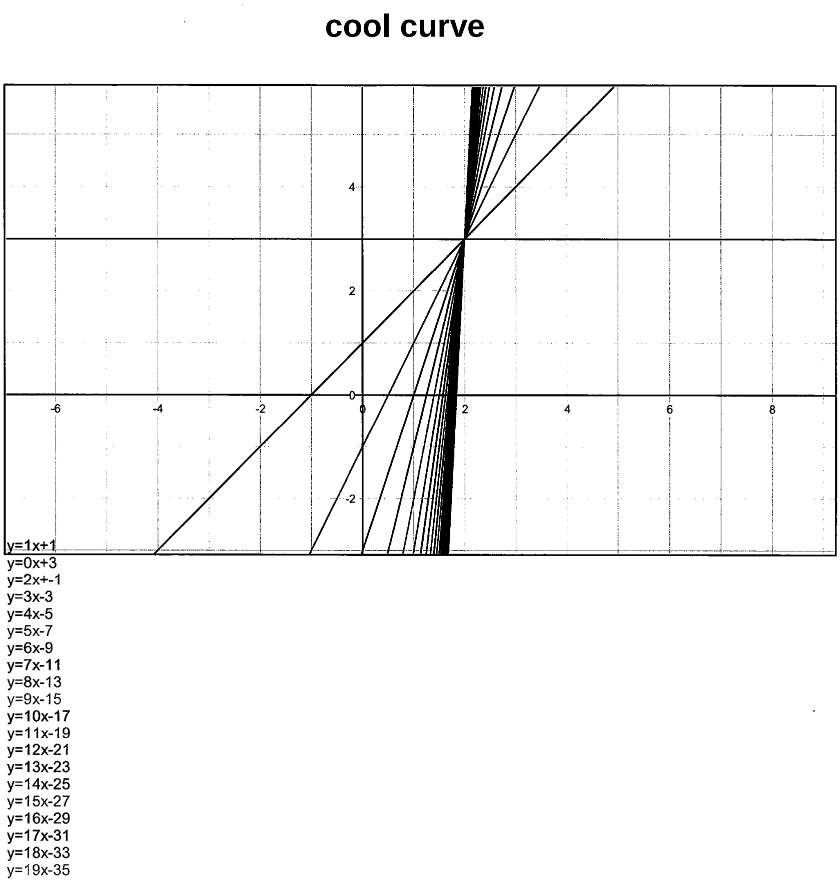


FIGURE 6

The final task was intended to develop students' ability to generalise (Mason, 1996). I asked them to make a similar drawing but with all their lines passing through a *different* common point of their choice. It was easy for students to differentiate this task for themselves (Watson, 1995) by choosing a point with non-integer coordinates (harder) or the origin (easier), depending on what they felt comfortable with attempting. Some tried to alter the equations that they had found for $(2;3)$ to translate the whole thing to another point - with mixed success! Most could now use their knowledge of the significance of m and c to find the lines they wanted much more quickly, and all discovered interesting patterns in their formulae.

Lines passing through the common point $(p;q)$ will satisfy the equation $y-q=m(x-p)$, giving $y=mx+(q-mp)$, so the c value for a given m value must be given by $c=q-mp$. For other ways of using graph-drawing software to create pictures, see Foster (2011).

Making connections from the symbolic/algebraic to the visual/geometrical is vital for students' subsequent mathematical development. Tasks such as these ones exploit ICT to confront students repeatedly, and instantly, with the visual result of their algebraic input, allowing them to make adjustments and see the consequences directly. A requirement such as that the lines must pass through a particular point, though with no specification regarding the angle, gives students a definite objective (constraint) amid a great deal of flexibility (freedom). In this way, students can explore possibilities without descending into simply producing random lines all over the screen. Having an aim for how the lines should be focuses students' attention on the details of the image and how to make modifications to the line by altering the equation. Such ways of working enable students to build up strong links between the algebraic and the geometrical and provide a firm foundation for future study.

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