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Less is More: Improving by Removing

(The 2023 Presidential Address)

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Introduction

The pressures currently on teachers of mathematics at every level seem greater than ever, whether that is university mathematics lecturers, school teachers or those who work with very young learners. So, I began my Presidential Address with the hope that no one would leave the *Joint Conference of Mathematics Subject Associations 2023* feeling as though they were not good enough and must do more. I did not want anyone to go home with a lengthy to-do guilt list of additional things they must try to incorporate into their practice. Instead, I advocated giving attention to the ancient proverb "less is more", versions of which are found within many cultures; e.g., "brevity is the soul of wit" (Hamlet), “Sometimes diminishing a thing adds to it; Sometimes adding to a thing diminishes it” (Tao Te Ching). I offered five aspects of mathematics teaching and learning that might benefit from ‘less of’ something, in each case trying to see how this might create space for more of something else. In this article, I present these five suggestions for *improving by removing*.

According to a recent paper in *Nature*, when asked to solve a variety of kinds of problems, people tend to default to proposing additive transformations, and they systematically overlook possible subtractive changes that could be made [1]. Just as a doctor may be inclined to prescribe supplementary pills to deal with the side-effects arising from previous ones, rather than discontinuing one of those previous pills, teachers may find it harder to notice places where subtracting something from our practice may be more beneficial than adding anything new. However, it is well worth spending the time needed to find these opportunities for removal. A first draft of a piece of writing can often be improved by deleting unnecessary words, making it more concise and easier on the reader. As Mark Twain is supposed to have written, “I didn't have time to write you a short letter, so I wrote you a long one instead.” It does indeed take effort to remove things, but investing in subtractive solutions seems potentially valuable. So, here are five suggestions for *doing less but doing it better*.

1. *Less teacher designing from scratch; more adapting of what already exists*

I think that 10 years ago this first suggestion might have been quite controversial, but it is certainly less so now. At Loughborough University, I direct the Loughborough University Mathematics Education Network [2], which provides completely free, high-quality, research-informed continuing professional development for teachers of mathematics at all levels, from Early Years up to university. (Please do take a look at the resources available at https://www.lboro.ac.uk/lumen/.) Our focus currently is on developing a complete set of free mathematics teaching resources for Years

7-9 (ages 11-14), and one feature of these is that they are entirely editable, under a *Creative Commons* licence. This means that teachers and schools can pull out any parts that they wish to use or adapt, as well as simply adopt the entire curriculum as it is, if they prefer [3]. We are really keen for teachers to modify and improve on the materials and tailor them to their teaching styles and to the learners that they teach. We feel that there is no need for teachers all over the country to be reinventing the wheel by starting from scratch.

Once the shift is made from designing from scratch to adapting something that already exists, it is possible to focus more carefully on the details. One of my Presidential blogposts was about ‘butterfly effects’ in mathematics task design, and how sometimes apparently superficial changes to a task can materially affect how it is experienced by learners. (I also gave an MA webinar on this topic in October 2022.) The devil is in the detail in educational design, but it is hard to think about these details when overwhelmed with the big task of designing everything from scratch.

When I had the privilege of working with Professor Malcolm Swan at the University of Nottingham, I learned that, whenever possible, you should avoid beginning with a blank sheet of paper. Try to begin with something that is *already* good – and improve it. Often, in a design meeting, Malcolm would reach for something from his extensive shelves of mathematics resources, which we could take as a starting point. How could we build on this and make it better? Of course, doing this means that the challenge then becomes making sure that you do not ruin what you begin with – i.e., avoiding ‘lethal mutations’! The key to that is in making sure that you understand what you are starting with, before trying to change it. Malcolm used to advise teachers, when using a new resource for the first time, to try to use it ‘as intended’, and see what happens. Only from the second time, is it advisable to start changing things. You might be surprised on that first run by what happens; for example, aspects that looked as though they would be too difficult for your learners might turn out not to be.

An enormous time-save for me in lesson planning was when I decided to stop planning *lessons*. By this I certainly do not mean ‘stop planning’ altogether! I mean stop planning individual lessons as set pieces, with a beginning, middle and end. Instead, plan a long ribbon of activities, carefully sequenced, but without worrying much about where the lesson breaks will occur. A teacher who seeks to be truly responsive to what happens in their classroom cannot really know how far they are going to get by the end of any particular lesson. And this can mean that teachers feel that they cannot plan Wednesday’s lesson until they see what happened on Tuesday – resulting in lots of late evenings spent doing last-minute planning and re-planning. I think there is very little benefit to doing this. I remember being told while I was training never to start something new in the last five minutes of a lesson. This kind of thinking led me to have lots of ‘fillers’ up my sleeve for spare moments, which in the end often meant wasting time, doing extra things that I did not value enough to have planned into my sequence. Now, I would happily start something new in the last five minutes. This can provide an opportunity to raise some questions, get learners thinking about something, and then, in the next lesson, you can ask them to recap this. Similarly, breaking a 20-minute activity half way through because of a lesson break can be productive, with everyone coming back to it with fresh eyes and ideas. There is never a bad time for a lesson to end, so I think there is no need to try to make everything fit neatly – learning is messy, and we cannot avoid that.

2. *Less crowding of content; more breathing space for sensemaking*

Many people seem to think that our mathematics national curriculum in England is overcrowded, and university teachers also frequently complain about the time constraints of fitting required content into a limited number of sessions. It is sad to see a teacher share a resource or a lesson idea with another teacher, and hear that teacher respond, "That's a nice idea. I'd like to do that. But unfortunately we only have space in the curriculum for two lessons on Pythagoras's Theorem." Clearly, we must prioritise, but it is too easy to answer the question: *What are the things that really matter in the mathematics curriculum*? Every teacher could give a long list, and the more time they had the more things they would include. But the harder question actually matters more: *What are the things that are of less importance*? Unless we can answer this, we will be trying to prioritise everything, which is the same thing as prioritising nothing. Although we may feel some affection for everything in mathematics, a typical list of curriculum topics will contain very different kinds of things that should have very different levels of importance. I think it should not be controversial to say that Pythagoras's Theorem is more important than, say, box plots. We need to give a good amount of time to the really important things, and that cannot be everything. It is better to do some topics properly, even if that means rushing others, than to just end up rushing through everything, doing nothing in depth.

Within the LUMEN Curriculum that we are developing at Loughborough, we see $y = mx$ as absolutely fundamental to the entire Year 7-9 (age 11-14) curriculum, and so we try to relate everything to straight lines through the origin whenever we can. We hope that this unifying principle will make learning more efficient, because topics that might seem superficially different are encountered as 'just another example' of the same thing. For instance, it is common to write $y = kx$ in the context of proportionality, but using a letter k may obscure the fact that this is identical with the multiplier, m, which is the gradient of our straight line through the origin. Even the '$+c$' in $y = mx + c$ is a distraction. Having a '$+c$' simply means that we have chosen the 'wrong' origin. So, we have decided to leave the introduction of the '$+c$' until much later (perhaps Year 9), after the big idea of $y = mx$ has been experienced deeply across many contexts. When the '$+c$' does make its appearance, we prefer to bring it in as $y - c = mx$. The origin was in the wrong place, so we moved it. Instead of seeing

$y = mx + c$ as an example of *non*-proportion, we prefer to see $y - c = mx$ as just another example of *proportion*, but with a different (transformed) dependent variable [4].

I have expanded on our emerging approaches to multiplicative relationships in another Presidential blogpost. Learners work within the context of a scenario such as the one shown in Figure 1. We see how the multiplier m can be between two variables (Figure 2), in which case we call it a *rate*, which may have units, or it can be *within* the same variable, in which case we call it a *scale factor*, which is always dimensionless (Figure 3). We represent a variety of scenarios, always using the Cartesian graph representation – initially, drawing the graphs accurately, and then later on drawing them schematically (i.e., not to scale). Learners explore answering questions both using rates and using scale factors, and begin to notice when the particular numbers in the question make one of these easier than the other.

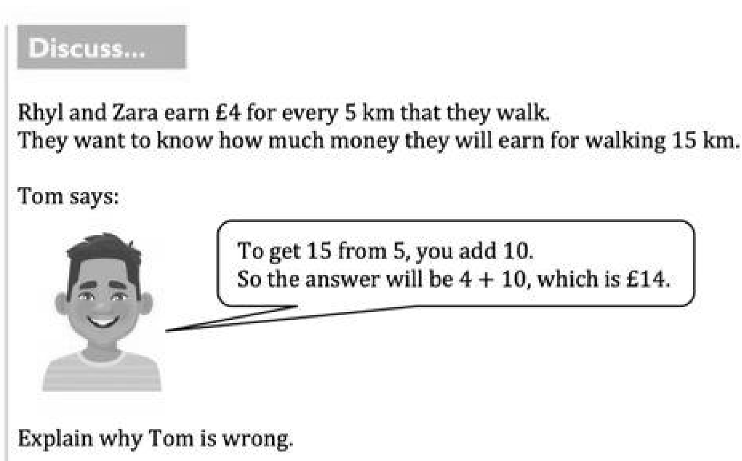


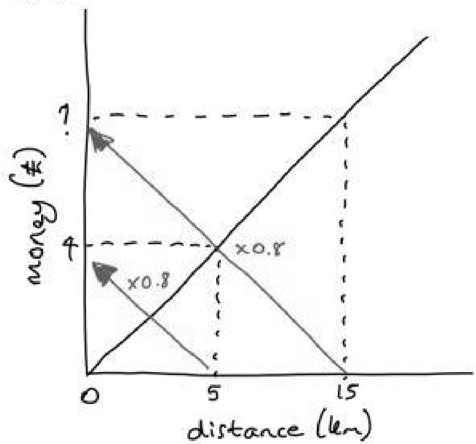
FIGURE 1: The initial scenario

It seems to us that big ideas do not come much bigger than 'straight lines through the origin', and we have written elsewhere about how $y = mx$ connects to rearranging formulae such as speed-distance-time and a unit-circle approach to trigonometry.

We also hope that having fewer, bigger ideas like $y = mx$ could, as well as deepen learners' connected understanding of mathematics, create breathing space for reasoning tasks that are not focused on specific areas of new content.

Discuss...

Rosie disagrees with Tom's answer.
Instead, she sketches a graph:



How did Rosie work out the 0.8 on the blue arrows?
Why does she write a multiplication sign in front of this?
What answer does Rosie obtain?
Do you think her answer is correct?

FIGURE 2: A between-variable multiplier, which we call a 'rate'

For example, consider this task:

Task 1

Asim owes Bel £305.

Bel owes Asim £307.

Asim and Bel want to pay their debts, so they visit a cash machine together to draw out the necessary money.

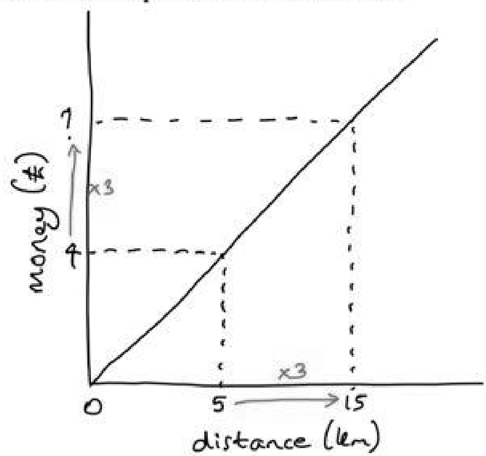
But the cash machine won't allow them to draw out that much money.

What should they do?

Learners will quickly realise that £300 of these amounts (and perhaps £305) 'cancels out', so Bel just needs to give Asim £2. This is just a warm-up to ease into the context.

Discuss...

Xena suggests a different way.
She rubs out Rosie's blue ink and puts her idea in red ink.



How did Xena work out the 3 on the red arrows?
Why does she write a multiplication sign in front of this?
What answer does Xena obtain?
Do you think her answer is correct?

FIGURE 3: A within-variable multiplier, which we call a 'scale factor'

Now consider Task 2.

Task 2

Asim owes Bel £305.
Bel owes Chloe £307.
Chloe owes Asim £100.
Find a way for them to settle up, so that no one owes anyone anything.
What is the easiest way for them to settle up?
Devise a method for solving 'settling up' problems like this.

This is a much more difficult task, but still depends on essentially 'zero' prior knowledge. There is no need to test learners for necessary pre-requisites before doing this task. It can be hard to find space in a crowded curriculum for 'no-priors' tasks like these, because they may not seem to be advancing in an obvious way the list of things that learners need 'to be able to do'. However, such tasks can be enormously important opportunities for developing mathematical reasoning. They also tend to have a levelling effect, because everyone has to think, and anyone can shine. A learner who has been absent recently, or who does not often feel successful in mathematics lessons, might be the star in this one. Anyone can learn from anyone, and, importantly, anyone can devise a problem that everyone else will find challenging.

With n people it will always be possible to settle up in $n - 1$ transactions, rather than n. But sometimes we can do even better than that. How can you devise the numbers so that three people can settle up in just 1 transaction? (See Yao [5] for more details.)

3. *Less rote memorising; more connecting of important ideas*

I do not think that rote memorising is always a bad thing. However, I have been thinking about this in the context of the multiplication tables, and the requirement for children in Year 4 (ages 8-9) to complete a timed multiplication tables check up to 12×12. Treating this as a memory test of 144 facts misses the point, I think, and completely fails to capitalise on the structure of these facts. Learning the multiplication tables should not be approached like learning something like the first 144 digits of π.

Some facts are indeed hard to reconstruct quickly. The fact $7 \times 7 = 49$ is a good example. It seems to me that this is hard to arrive at from other facts that are likely to be known if this one is *not* known. And so I think there is a good case for learning $7 \times 7 = 49$ by rote. But 7×7 is the exception, rather than the rule, and the vast majority of the multiplication facts are so intimately connected to each other that it seems to me absurd to think of them separately, which I think is often what happens in practice.

Usually, when people teach the tables facts in a more *connected* way, they do it 'within table', by, for example, learning all of the 6-times tables at once. For me, this inevitably becomes 'additive', with learners working out something like 6×8 by doing, say, $6 \times 10 - 6 \times 2$. In my free little booklet *Learning Times Tables Through Systematic Connections* [6], I avoid this by suggesting teaching the facts in an order that better respects their multiplicative structure, and never involves any addition or subtraction. The approach involves learning by rote just the nine facts: 3^2, 4^2, 6^2, 7^2, 8^2, 12^2, 3×4, 3×7 and 4×7. From these, all of the remaining facts can be obtained extremely quickly by simple transformations – often doubling, in which no carrying is involved – and I set out the details in the booklet. Less memorising is definitely *more* here. This is not only more efficient use of time but builds important skills of 'derived facts', enabling learners to find products outside of the 12 × 12 tables as well. For example, as well as obtaining 12 × 8 from doubling 6 × 8, learners can just as easily obtain 6 × 16, or 3 × 32. It is intended to be a more 'sense-making' approach.

For me, the unscalability of rote learning the multiplication tables reveals its limitations. In the Presidential Address, I asked the audience to consider how they would prepare for a test of *their* own times tables knowledge – up to 20 × 20, which I anticipated most participants would not be fluent in recalling. At first glance, it might seem that if you already know up 12 × 12 then there are just another 8 tables to master, so you are more than half way there. However, in fact, $20^2 - 12^2 = 16^2$, so there are more new facts to grasp than you already have. Books such as Talwalkar [7] offer 'tricks' based on the fact that $(10 + a)(10 + b) = 100 + 10(a + b) + ab$, which allows us to calculate a product such as 14 × 18 very quickly without paper:

1. 4 + 8 = 12
2. 12 × 10 = 120
3. 120 + 100 = 220
4. 220 + 4 × 8 = 252

This is far more enjoyable and insightful than *memorising* (and almost certainly muddling up) a lot of new products, and I cannot imagine why anyone would seek to *memorise* the tables up to 20 × 20.

4. *Less calculating; more estimating and reasoning*

Children's journeys into mathematics inevitably begin with counting and calculating. When adults seek to 'be mathematical' with young children, they invariably ask them, "How many are there?", aiming to get children interested in and fluent with counting. As learners progress through school, they are increasingly asked to count and calculate things, almost always for little purpose. (Imagine a child responding: "Supposing we knew how many stairs there were here – how exactly would that help us?") In a world full of technology, which possesses calculating skills no human can ever hope to compete with, a shift away from counting and calculating towards estimating and reasoning seems highly overdue. One way into this is the question "Which is less or are they the same?", which I find that I come back to again and again.

Being mathematical often means avoiding counting – certainly 'counting all' should always only ever be a last resort. To decide "Which is less or are they the same?" in Figure 4, we need only to be able to count up to 4 (not 13), because we can discern a common 3 × 3 block, and the left-hand collection has 3 more than this, and the right-hand collection has 4 more (Figure 5). In this case, we not only discover which is less but *how much less*. The whole purpose of multiplication is to avoid counting, and we should always avoid counting whenever possible in favour of less mundane and more creative reasoning. This is one application of the 'mathematicians are lazy' mantra.

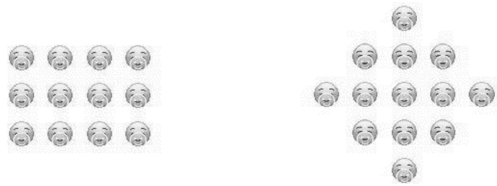


FIGURE 4: Which is less or are they the same?

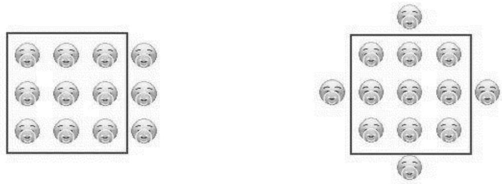


FIGURE 5: Seeing the solution using counting only up to 4.

The other non-calculation approach that I think is much neglected in school mathematics is estimation. In an ideal world, we would always estimate an answer before calculating, and especially before using technology to calculate for us, so that we can make intelligent use of the machine and notice when it is giving us nonsensical answers. (In the lecture I showed an image of the QAMA calculator - http://qamacalculator.com/ - which cleverly forces this practice on learners. Before it will give you an accurate answer to a calculation, you must give it a reasonable estimate!)

The question "Which is less or are they the same?" can be applied to all sorts of mathematical content, and with mathematics of varying degrees of complexity. In none of these tasks is there any need to get your hands dirty with actual calculations!

Task 3

Which is less or are they the same?

6×8 or 7×7

16×18 or 17×17

1116×1118 or 1117×1117

Are these questions getting harder?

Task 4

Which is less or are they the same?

6×9 or 7×8

16×19 or 17×18

1116×1119 or 1117×1118

Are these questions harder than those in Task 3?

One of my favourite tasks is:

Task 5

Which is less or are they the same?

36×48 or 46×38

One approach to this task is to consider 36 by 48 and 46 by 38 rectangles. What is the same and what is different about these? They have the same perimeter, but different areas. This leads to the observation that, among a set of equal-perimeter rectangles, to locate the one with the largest area we need the one ‘closest to a square’. This means that we want our two dimensions to be as close to each other as possible, so the smaller 10s digit needs to be partnered with the larger 1s digit. This works because the gap between 38 and 46 is smaller than the gap between 36 and 48 (and we see this *not* by calculating these subtractions, but by observing that 36 increases to 38 and 48 decreases to 46). Similar reasoning is the key to solving this task:

Task 6

Use the digits 1 to 9 to make two numbers which multiply to give the greatest possible product.

Some other “Which is less or are they the same?” tasks that I shared during the Presidential Address are given below. Before you solve them, try to see whether you have a gut feeling one way or the other.

Task 7

Which is less or are they the same?

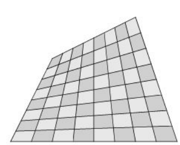
1. A 5% increase followed by another 5% increase, or a 10% increase.
2. $\sqrt[9]{9}$ or $\sqrt[10]{10}$
3. $\sqrt[9]{9!}$ or $\sqrt[10]{10!}$
4. π^e or e^π

Task 8 [8]

Which is less or are they the same?

The total black area and the total white area on an ordinary chessboard.

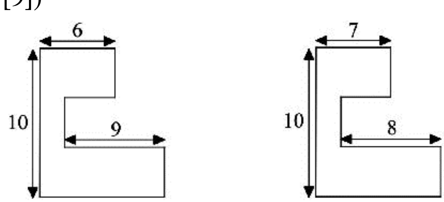
If that seems too easy, what about the *skewed* chessboard shown here?



Task 9 (adapted from Twitter [9])

Which is less or are they the same?

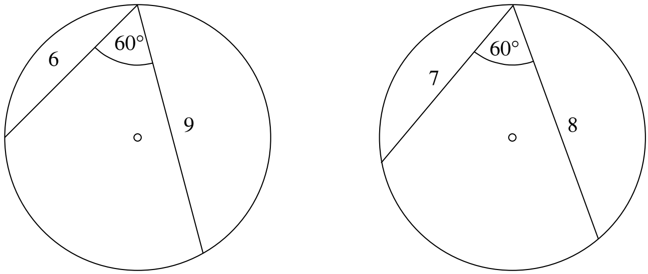
The perimeters of these two rectilinear figures.



Task 10 (adapted from Twitter [10])

Which is less or are they the same?

The radii of these two circles (not drawn to scale).



A nice solution to Task 7 #3, due to Posamentier and Salkind ([11, p. 157]), involves raising both roots to the 90th power:

$$\left(\sqrt[9]{9!}\right)^{90} \overset{?}{<} \left(\sqrt[10]{10!}\right)^{90}$$

$$(9!)^{10} \overset{?}{<} (10!)^{9}$$

$$(9!)^{9}(9!) \overset{?}{<} (9!)^{9}\,10^{9}$$

$$(9!) \overset{?}{<} 10^{9},$$

which is clearly true, meaning that $\sqrt[9]{9!} < \sqrt[10]{10!}$. (Incidentally, I am quite fond of the $\overset{?}{<}$ notation to mean, "*Is it* less than?". It can always be formalised into a proof by contradiction if desired.) Another approach with this one could be to see the 9 and 10 as 'arbitrary'. So, could we just observe that, say, $\sqrt[1]{1!} < \sqrt[2]{2!}$, since $1 < \sqrt{2}$, and therefore conclude that $\sqrt[9]{9!} < \sqrt[10]{10!}$? That depends on being sure that the function $y = \sqrt[x]{x!}$ is monotonic. Is that obvious? It can be dangerous to argue this way. For example, with Task 7 #4, we might realise that $2^3 < 3^2$, and $2 < 3$, which might suggest that $e^\pi < \pi^e$, since $e < \pi$. But in fact, this switches around, because $3^4 > 4^3$, even though $3 < 4$, so a convincing argument for which of e^π or π^e is less needs more thought than this.

I finished this section of the Presidential Address by exploring an undergraduate student's question when first being introduced to $i = \sqrt{-1}$. Their question was, "How big is i?" This struck me as a very reasonable question to ask about any purported new 'number'. To see some discussion of my approach to answering the student, go to my blogpost, and a video of this is available on Youtube [12].

5. *Fewer different representations, models and examples; more depth*

I will be brief here, partly because this may be where I venture into the most controversial of my five suggestions, but mainly because I have discussed this at length elsewhere. In our design work on the LUMEN

Curriculum, mentioned above, we have chosen to prioritise number lines and Cartesian graphs (which we view as two number lines intersecting at 90°). While other representations would undoubtedly bring additional insights, I suggested in the Address that our curse of knowledge might lead us to think that things become clearer when we look at them in lots of different ways. This may be true for us as teachers, but, for learners, comparing multiple representations, models and examples may just make things seem even more confusing. The *representation dilemma* [13] draws attention to the cost of introducing every new representation or model, in terms of getting learners sufficiently fluent with it for them to be able to benefit from its insights. And this time and energy could always instead be spent on becoming even more confident with already-known representations and seeing their wider applicability. So far in the LUMEN work, we are seeing that you can really do a very great deal with just number lines and Cartesian graphs!

Conclusion

All of us certainly need to change things in our practice, otherwise "learning from experience may lead to nothing more than learning to make the same mistakes with increasing confidence" [14]. We never get beyond learning and improving our practice. However, change does not necessarily have to mean doing more each year and escalating our workload to dangerous levels. Perhaps we all should consider what we might be *improving by removing*?

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