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Triangular Roots

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This article defines triangular roots of real numbers by analogy with square roots. It shows that the triangular root of a real number can never be purely imaginary: it is either real or complex with a real part of $-\frac{1}{2}$.

We all know about *square roots*. For a nonnegative real number a, we say that b is a square root of a if $a = b^2$. When $a > 0$, it has two square roots and we denote the positive one by $\sqrt{a}$; when $a = 0$, there is just one root: zero. Thus the sequence of perfect squares is $1, 4, 9, 16, \dots$ and

$$1 = \sqrt{1}, \qquad 2 = \sqrt{4}, \qquad 3 = \sqrt{9}, \qquad 4 = \sqrt{16},$$

and so on. The nth perfect square is the number of dots in an $n \times n$ square array, as shown in figure 1.

Now can we define a *triangular root*? The nth triangular number t_n is defined by

$$t_n = 1 + 2 + 3 + 4 + \cdots + n = \tfrac{1}{2}n(n+1),$$

and is the number of dots in the first n rows of the triangular array shown in figure 2. For a real number a, we say that b is a triangular root of a, and write $b = \text{tr}(a)$, if

$$a = \tfrac{1}{2}b(b+1). \tag{1}$$

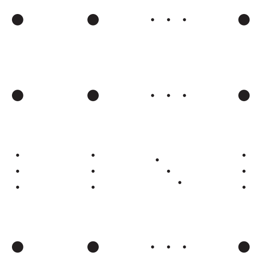

Figure 1

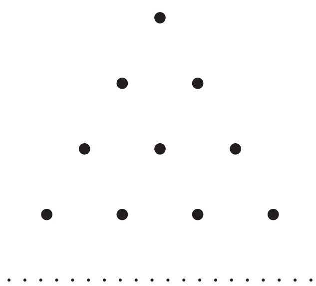

Figure 2

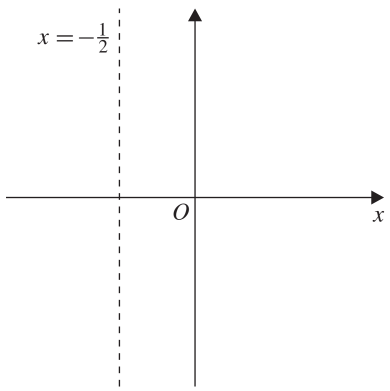

Figure 3

Thus, the sequence of triangular numbers is $1, 3, 6, 10, \ldots$ and

$$1 = \text{tr}(1), \qquad 2 = \text{tr}(3), \qquad 3 = \text{tr}(6), \qquad 4 = \text{tr}(10),$$

and so on.

From (1),

$$b^2 + b - 2a = 0,$$

which is a quadratic equation in b with roots

$$b_1, b_2 = \frac{-1 \pm \sqrt{1+8a}}{2}.$$

As with square roots, we take the positive sign to give

$$\text{tr}(a) = \frac{-1 + \sqrt{1+8a}}{2}.$$

This choice of sign agrees with

$$1 = \text{tr}(1), \qquad 2 = \text{tr}(3), \qquad 3 = \text{tr}(6), \qquad 4 = \text{tr}(10),$$

and so on, and provided that $a \geq -\frac{1}{8}$, it will have a unique real triangular root.

We know that $\sqrt{1+8a}$ is a real number when $1 + 8a \geq 0$ and a purely imaginary number when $1 + 8a < 0$ (e.g. $\sqrt{-4} = 2\text{i}$), and the square roots of real numbers lie on the real and imaginary axes. So if $a \geq -\frac{1}{8}$, the triangular roots lie along the real axis, whereas if $a < -\frac{1}{8}$ then they lie along the line $x = -\frac{1}{2}$ (see figure 3).

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