Being Inclusive

by **Colin Foster**

A hexagon has five sides. True or false? 'False', you might say – 'a hexagon has *six* sides'. But if it has six sides, then it certainly has five sides. If I have ten pound coins in my pocket and somebody asks me, 'Do you have two pounds?' then I should answer 'Yes', shouldn't I? If I didn't want to give them my money, and so answered 'No' on the grounds that I had *ten* pounds, not *two* pounds, you would think that a bit dubious, wouldn't you? To protest that 'They should have asked me whether I had *at least* two pounds!' sounds a bit hollow. So a hexagon has five sides, a square has three right angles, a rectangle has one line of symmetry and a triangle has two vertices.

I think this is how pupils sometimes feel when we ask them to accept 'inclusive definitions'. We show them a square and ask them if it's a parallelogram, and they say, 'No, it's a square, stupid', with the 'stupid' perhaps implied rather than stated. And we think, 'Oh dear. They don't realize that all squares are parallelograms'. But maybe they do but they are just trying to answer more accurately. They know that a hexagon has five sides but they think that it would be less misleading to say that it has six sides. So when you ask them if it has five sides they say 'No, it has six'. It is a different view of what we mean by 'precise'.

Generally, pupils are rewarded for giving more detail. In a languages lesson when the teacher shows vocabulary pictures, the pupils will not be praised for saying 'a thing' to every picture. They are expected to give the most detailed name they know, like 'bus' or 'mushroom'. And the same is true in mathematics – except when we are playing this strange 'inclusive definitions' game. In normal circumstances, if a mathematics teacher draws a regular hexagon on the board and asks 'What do we call this?', they will not be very impressed with the answer 'A shape'. They will sigh and say, 'Yes, what *kind* of shape? … What *kind* of polygon? … What *kind* of hexagon?' until they get to the most detailed description. 'Square' is easier to say, shorter and much easier to spell than 'parallelogram', so why not call a spade a spade and a square a square?

Being 'inclusive' sounds like a nice thing. We associate the word with schools that are open to all and don't have nasty entrance exams and fees to keep people out. We think of open-minded anti-discriminatory people who don't bully those who are different. But, for pupils, being inclusive with mathematical definitions seems to make things very hard. I remember a pupil getting quite distressed in a lesson when I said that squares were rectangles. 'You're basically telling me that everything I know is wrong. A packet of crisps is a banana but a banana is not an apple – this does my head in!' Her outburst had arisen because I had been saying that in everyday life a flower is a plant but a plant is not necessarily a flower; all maths teachers are teachers, but not all teachers are maths teachers – that kind of thing. I now think that the difficulty she was having was not really with the necessary/sufficient logic but with the existing images she had of the concepts: to her, a rectangle was what I would call an oblong – i.e. a non-square rectangle. So to tell her that a rectangle is a square is as daft as saying that 'a packet of crisps is a banana'.

When these shapes are first introduced, they tend to be presented in their 'standard' canonical forms. We all know what 'a parallelogram' is supposed to look like (even which way it should be leaning!), and when I say 'a parallelogram' I am sure that the image that pops into your mind is not a square. Textbooks and teachers even ask questions like 'How many lines of symmetry does a parallelogram have?', and they expect the answer 'None'. Well, they probably expect answers like 'One' or 'Two', but they think that the *correct* answer is 'None'. But really the correct answer is 'It depends what kind of parallelogram'. If it's an oblong parallelogram or a non-square rhombus parallelogram, then it has two lines of symmetry; if it's a square parallelogram, then it has four; otherwise, it has none. I have never seen that answer in the back of the book! These things are complicated and I think it is normal practice in schools to be inconsistent – to say, 'Oh, you know what I mean' one day but then suddenly switch and starting expecting pupils to use inclusive definitions.

Being inclusive mathematically is supposed to simplify things, because if we prove something for parallelograms, then because squares are parallelograms it should automatically apply to squares too. But this is not the case with lots of properties, such as order of rotational symmetry, number of equal sides/angles, and many others, which change from general parallelogram to general rhombus to general rectangle to square. This leads to all sorts of problems. For example, is a parallelogram a trapezium? This depends on whether a trapezium has *exactly one* pair of parallel sides or *at least one* pair of parallel sides. When visualizing integration by fitting trapezia underneath a curve (or using the ‘trapezium rule’), some of those ‘trapezia’ may well be rectangles, so it would be nice if we could call them all trapezia. And the area formula $\frac{1}{2}(a+b)h$ for a trapezium works for a parallelogram, by letting $a = b$, so it is very nice to be able to say that a parallelogram is an example – just a special case – of a trapezium. But a bit more thought reveals that the area formula for a trapezium also works for a triangle if you let $a = 0$ (imagine the shorter parallel side shrinking down to zero), so does that mean that a triangle is a trapezium in which one side happens to have zero length? Does this mean that all triangles are quadrilaterals and all pentagons are hexagons? Reversing the statement we started with, we could say that a pentagon has six sides – one of which just happens to be of zero length!

In the same sort of way, pupils are often asked to sort triangles into scalene, isosceles and equilateral, and these are initially presented as mutually exclusive categories. No triangle should get two of these names. But then later they may be told that equilateral is actually a subset of isosceles, because isosceles doesn’t, after all, mean ‘*exactly* two equal sides’ but ‘*at least* two equal sides’ – although the pupils are certain that they don’t remember being told that when they first met the word! It feels like the rug has been pulled out from underneath them. A pupil once asked me whether that meant that scalene triangles had ‘at least one equal side’, and therefore that isosceles triangles were a subset of scalene triangles. Pupils have a keen sense of symmetry and completeness, even though neither of us could say what ‘one equal side’ might mean – equal to what? ‘Itself’ was the answer the pupil gave me.

Sixth-formers sometimes complain that in science they were taught a model of the atom for GCSE that they are now told was complete nonsense and they have to learn a new quantum-mechanical model. This may be a little unfair on their science teachers, but do we perhaps do the same thing in mathematics? Do we give definitions like ‘An isosceles triangle has two equal sides’ with our fingers crossed behind our back because we know that it could actually have three equal sides? Keeping things ‘simple’ to avoid confusion is often a recipe for confusion. Pupils should not have to worry about ‘what we are doing today’ in order to know what kind of answer is right, yet it does seem as though they have to switch inclusive definitions on and off as a result of often quite subtle cues. If a pupil tries to be clever when answering ‘How many fingers am I holding up?’, they are going to get labelled as low-attaining, or worse!

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