Differences over Differences Methods:

Pros and Cons of Different Ways of Finding the *n*th Term of a Sequence of Numbers

by Colin Foster

A quick informal survey among colleagues revealed a number of different approaches to finding an expression for the nth term of a sequence of numbers. We give a lot of attention to this particular skill because of its usefulness in the GCSE coursework investigation, so it seemed worth considering the pros and cons of the different methods.

Method 1. Inspection

The quickest approach to many sequences, though the hardest to learn, is the "Ah, I see what it is!" method! Sometimes the clue comes from the source of the sequence, perhaps the geometry of a pattern, for example. Sometimes, it is possible to see how numbers with not too many factors might have been produced.

n	1	2	3	4	5
u	8	15	24	35	48

For example, in this sequence, 15 suggests itself as 3×5, and 35 as 5×7. Checking this pattern for the other terms quickly confirms that $u = (n + 1)(n + 3) = n^2 + 4n + 3$.

Pupils frequently need to be encouraged that inspection is a 'proper method' and not 'cheating', although they'll need to explain that it wasn't simply inspection of someone else's work!

Method 2. Simultaneous Equations

All the other methods begin by finding the difference between successive values, and then the differences between those successive differences, and so on until a row of differences turn out to be constant. For example,

n	1		2		3		4		5
u	8		15		24		35		48
1st differences		7		9		11		13	
2nd differences			2		2		2		

If the 1st row of differences turn out to be the same, the sequence is 1st degree (linear): $u = a + bn$.

If the 2nd row of differences turn out to be the same (as above), the sequence is 2nd degree (quadratic): $u = a + bn + cn^2$.

If the 3rd row of differences turn out to be the same, the sequence is 3rd degree (cubic): $u = a + bn + cn^2 + dn^3$.

And so on, where a, b, c, d, etc. are all constants to be determined. (The final constant in each equation can't be zero, otherwise the sequence wouldn't have the required degree, but some or all of the other constants could be zero.)

The simultaneous equations method starts with the appropriate formula (quadratic for our example), and with three unknowns to find (a, b and c) substitutes n and u values for the first three terms, as below.

$$u = a + bn + cn^2$$
$$n = 1 \quad 8 = a + b + c$$
$$n = 2 \quad 15 = a + 2b + 4c$$
$$n = 3 \quad 24 = a + 3b + 9c$$

Then the task is to solve these simultaneous equations to find a, b and c and hence the formula for u.

The main difficulty with this method is solving these simultaneous equations, which for a cubic (or higher) sequence can be heavy work. GCSE pupils are not normally expected to solve simultaneous equations in more than two unknowns, and three is necessary even for a quadratic sequence. For that reason I don't generally encourage pupils down this route.

Method 3. Comparison with the General Case

This begins by finding differences until a row of constant differences are obtained (as above) so that the degree of u is found. Then the differences method is applied to the *general* case of that degree, as below for a quadratic sequence $u = a + bn + cn^2$.

n	1		2		3		4
u	$a+b+c$		$a+2b+4c$		$a+3b+9c$		$a+4b+16c$
1st differences		$b+3c$		$b+5c$		$b+7c$	
2nd differences			$2c$		$2c$		$2c$

By comparing the *numbers* in the differences table for our sequence with the *expressions* in this table we can say that

$$2c = 2 \Rightarrow c = 1$$
$$b + 3c = 7 \Rightarrow b + 3 \times 1 = 7 \Rightarrow b = 4$$
$$a + b + c = 8 \Rightarrow a + 4 + 1 = 8 \Rightarrow a = 3$$

(using the left hand positions on each line of the table). So $u = 3 + 4n + n^2$ or $n^2 + 4n + 3$.

Solving these equations is always easy provided you work from the bottom row of the table upwards. I have become very fond of this method, and pupils can use it to find the nth term of cubic, quartic or higher sequences without much difficulty.

There is a variant of this method, where the differences table is extended to the left to find the 0th term and a column for $n = 0$ is put into the table. (This is done by working from the bottom line upwards.)

So we get these tables.

n	0		1		2		3
u	a		$a+b+c$		$a+2b+4c$		$a+3b+9c$
1st differences		$b+c$		$b+3c$		$b+5c$	
2nd differences			$2c$		$2c$		$2c$

n	0		1		2		3
u	***3***		8		15		24
1st differences		***5***		7		9	
2nd differences			***2***		2		2

Again, by comparing numbers with expressions,

$$2c = 2 \Rightarrow c = 1$$
$$b + c = 5 \Rightarrow b + 1 = 5 \Rightarrow b = 4$$
$$a = 3.$$

This is perhaps slightly easier, but requires finding additional numerical values, and mistakes can be made, particularly if any of the differences are negative. (The 0th term is very handy for finding a, the constant in the formula for u, the last line of working above.)

Method 4. Brackets Method

Here, differences again tell us what degree of equation is required. But then instead of writing a quadratic (say) as $u = a + bn + cn^2$, we instead use the equivalent form $u = A + B(n–1) + C(n–1)(n–2)$. The pattern would continue for, say, cubic, which would be

$$u = A + B(n–1) + C(n–1)(n–2) + D(n–1)(n–2)(n–3).$$

The advantage of this type of formula, is that substituting in the values of u and n is easy. No simultaneous equations need solving, because we can find $A, B, C, \ldots$ one at a time in that order.

For example, using our quadratic sequence,

$$u = A + B(n-1) + C(n-1)(n-2)$$
$$n = 1 \qquad 8 = A$$
$$n = 2 \qquad 15 = A + B \times 1 = 8 + B \Rightarrow B = 7$$
$$n = 3 \qquad 24 = A + B \times 2 + C \times 2 \times 1 = 8 + 7 \times 2 + 2C$$
$$= 22 + 2C \Rightarrow C = 1$$
So $u = 8 + 7(n-1) + (n-1)(n-2)$.

The substituting requires care, and the end result will need expanding and simplifying, which can be a big disadvantage of this method. But the strength is that a correct (though unsimplified) formula is obtained quickly.

Simplification is quickest by doing one bracket at a time:

$$u = 8 + 7(n-1) + (n-1)(n-2)$$
$$= 8 + (n-1)(7 + n - 2)$$
$$= 8 + (n-1)(n+5)$$
$$= 8 + n^2 + 4n - 5 = n^2 + 4n + 3$$

[For Newton's variation on this, see Wakefield (2003).]

Again it is possible to use this method with a differences table that goes back to $n = 0$ (see Method 3). Then the most convenient formula is $u = A + Bn + Cn(n–1)$ for quadratic, with similar formulas for cubic or higher sequences.

Method 5. Reducing the Degree by One

A long-winded but easy-to-understand method is first to find the coefficient of the highest power of n and then evaluate and subtract that term from each term of the original sequence to leave a sequence that is at least one degree less.

For example, constant 2nd differences of 2 tell us that in $u = a + bn + cn^2$ the value of c is 1. So we work out cn^2 ($= n^2$ here) for each value of n and subtract that much from each corresponding value of u.

n	1	2	3	4
u	8	15	24	35
n^2	1	4	9	16
$u - n^2$	7	11	15	19

The new sequence for $u - n^2$ is linear, and by differences or inspection the nth term is $4n + 3$, so $u - n^2 = 4n + 3$ or $u = n^2 + 4n + 3$.

The logic of this can be appealing, but it is too lengthy for cubic or higher sequences, particularly if some of the numbers are not integers.

Conclusions

For myself, I firmly reject Method 2. Method 5 may be helpful for understanding but is too inefficient for general use. Method 4 may be quickest if all you need is, say, the 100th term, because you can substitute $n = 100$ without needing to expand the brackets and simplify. But if you want an answer in the form $u = a + bn + cn^2 + \ldots$ I don't think you can beat Method 3.

That's just my conclusion. I'd be very interested to hear other readers' opinions.

Reference

Wakefield, P. 2003 'Newton's Forgotten Formula', *Mathematics in School*, **32**, 2.

Keywords: nth term; Differences method; Polynomial sequences.

Author
Colin Foster, King Henry VIII School, Warwick Road, Coventry CV3 6AQ.
e-mail: **colin@foster77.co.uk**