Questions Pupils Ask!

Doubly Positive

by **Colin Foster**

We weren't working explicitly on directed numbers at the time, but during a mathematics lesson a pupil suddenly asked me, seemingly out of the blue, '*If two minuses make a plus, why don't two pluses make a minus?*'

Perhaps this question had been on his mind for some time and he had only now found an opportunity to ask it. This caused me to think about the frantic nature of mathematics classrooms a lot of the time, with the teacher rushing around from pupil to pupil, dealing with urgent enquiries and desperate to fulfil their lesson objectives. Unless there is some slack in the lesson and an absence of pressure, perhaps many pupil questions are unlikely ever to be asked. Being 'focused' has its dangers.

I thought about what the pupil might mean by this. Did he mean 'two minuses' in the additive sense of 3 – (–4) or the multiplicative/division sense of (–3) × (–4) and (–3) ÷ (–4)? Do 'two minuses make a plus' in both of these cases for the same reason or for different reasons? The same convenient mantra 'works' for both, but they seem to be quite distinct situations. Does his language of 'pluses' and 'minuses', rather than 'positives' and 'negatives', perhaps indicate one scenario rather than the other?

There are plenty of good ways of seeing that 'two minuses make a plus' (French, 2001), but that doesn't really seem to be what the pupil is asking for. He might already be quite convinced that 'two minuses make a plus', and have thought of his question *because* of that. It can be tempting in the classroom, when asked a question, to jump to something you are comfortable with that may not really address it very well. Some commercial websites have automated helpdesks for answering customers' questions, and when you type in your question they offer you several possible questions together with their answers. But although all of their questions contain some of the same words as those in your question, it is often the case that none of them really answers what you are asking.

The pupil's question seemed to demonstrate a deep commitment to symmetry – the idea that what works for negatives should work in the same way for positives. This seems quite reasonable. When we had worked with directed numbers in class, we had used the model of positive and negative charges (actually, particles and antiparticles which annihilate each other) (Note 1). When pupils (at about this age) study electricity in science, there is a perfect symmetry between positive and negative charges; their labelling is quite arbitrary – a historical accident (Note 2). It would be unthinkable within school electrostatics for positive charges to do something that negative charges don't do: Maxwell's equations are invariant under charge conjugation (but see Note 3). So it is easy to see why the pupil might be disturbed by the idea that there is a breaking of the symmetry between positive and negative numbers.

Rather than switching to a different model, such as number lines, I thought it would be more helpful to remain with the electrical model which we had used previously. If we represent +1 by ⊕ and –1 by ⊖, then 'taking away' –4 from 3 is tricky, because when I draw ⊕⊕⊕ = 3 I can't see four negatives to remove. However, I may realize that this drawing is equivalent to

⊕⊕⊕⊕
⊖ = 3 or, more usefully, ⊕⊕⊕⊕⊕⊕⊕
⊖⊖⊖⊖ = 3

because positives and negatives have been added in pairs, and so cancel each other out. So any of these diagrams might be used to represent 3, but the final one allows some crossing out,

⊕⊕⊕⊕⊕⊕⊕
~~⊖⊖⊖⊖~~

which does seem to illustrate that when you 'take away' the 'four negatives' from '3' you get 7, so 3 – (–4) = 7.

It is clear in this model that ⊖⊖ is not the same as ⊕, any more than ⊕⊕ is the same as ⊖. In numbers, we have that (–1) + (–1) = –2, so 'two negatives' added together like this do not in any sense cancel each other out. It is a *positive and a negative* that cancel out, because $\oplus\ominus$ and $\ominus\oplus$ are both equal to zero. So when pupils say that ‘two minuses cancel each other out and become a positive’, perhaps this is potentially confusing. When we say that $3 - (-4) = 3 + (+4) = 7$, we are really saying that *subtracting a negative* is the same as *adding a positive*; more informally, ‘two minuses make *two* pluses’, and ‘two pluses make two minuses’, so perfect symmetry is preserved. We could re-label our positive and negative numbers the other way round and they would work in exactly the same way.

That seems reassuring, but what about multiplication and division? Since $(-3) \times (-4) = (+3) \times (+4) = +12$, the symmetry does seem to break down here. Positive numbers are those which, when two of them are multiplied together (or when they are squared), produce an answer of the *same* sign, whereas two negative numbers multiply to produce a number of the *opposite* sign to themselves. The symmetry has gone. What is going on here? Why does this happen? In physical terms, it isn’t meaningful to ‘multiply two charges together’, but the signed values of charges *are* multiplied together in physics when calculating the *force* exerted by one charge on another (Coulomb’s law). ‘Like’ charges produce a positive (repulsive) force, and ‘unlike’ charges produce a negative (attractive) force (Note 4). So what is the status of this ‘same signs make a positive’ rule – is it a convention, a definition or a necessary result?

In a multiplicative context, positive numbers and negative numbers do behave differently, and there is no reason to suppose that two positives should multiply to make a negative just because two negatives multiply to make a positive. We have a different sort of symmetry, of the Z_2 kind – ‘like numbers positive; unlike numbers negative’:

Z_2	+	–
+	+	–
–	–	+

The symmetry operates not at the level of individual numbers but at the level of *pairs* of numbers. Is this a satisfying response?

Thinking about possible ways of responding to this pupil’s question led me to consider the multitude of different models available for working with directed numbers. To what extent do these models complement one another, or appeal to different pupils, and to what extent do they unhelpfully clash? Are pupils better off the more models they know about? Some models are clearly less precise than others. Sometimes teachers use the analogy that in the English language a double negative is a positive, which has some truth to it, even though ‘I am not unhappy’ does not quite mean that ‘I am happy’! Beard (2009) includes an account of a lecturer who commented that ‘it was peculiar that, although there are many languages in which a double negative makes a positive, no example existed where two positives expressed a negative’ (p. 104–105). An audience member, in ‘a dismissive voice’, was heard to call out ‘Yeah, yeah’. So perhaps two pluses can sometimes be a minus.

Notes

1. For a lesson plan based on the idea of using electric charge for teaching addition and subtraction of directed numbers, see Foster (2013).
2. It was Benjamin Franklin (1706–1790) who dictated which way round positive and negative charge would be labelled. (See **http://xkcd.com/567/** for an amusing take on this.)
3. A physicist would point out that *C-symmetry* (the symmetry of physical laws when you swap positive charges for negative ones) *is* broken in weak interactions.
4. It’s a bit unfortunate that the words ‘positive’ and ‘repulsive’ go together, but there doesn’t seem to be much we can do about this!

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