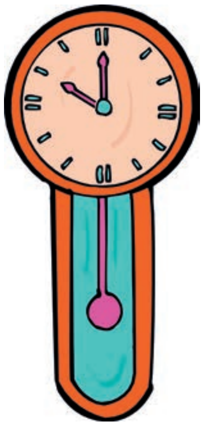
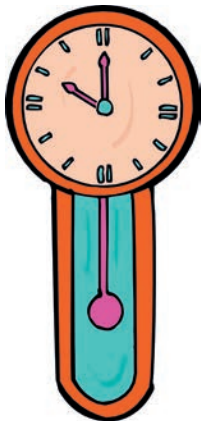


Family Negotiations



by **Colin Foster**

Sometimes I read something in a non-mathematical context that jumps out at me as potentially highly mathematical. Maybe the mathematics is 'in there' or maybe it's 'in me' – or maybe both? Here's an example from David Lodge's novel *Paradise News*:

> Roxy had gobbled down her salad and lemon casserole and departed with some friends to a drive-in movie. ("Don't be late," Yolande said to her, as a car horn cleared its throat in the road below the house, and Roxy leaped to her feet. "How late is late?" "Ten o'clock." "Eleven." "Ten thirty." "Ten forty-five," Roxy yelled from the porch, as the screen door slammed behind her. Yolande sighed and grimaced. "It's called family negotiation," she said.)
>
> (Lodge, 1991, p. 138)

The scenario is similar to financial negotiations in which the seller's opening position overstates how much they expect to get and in a similar way the buyer (if they are savvy) understates how much they are willing to pay, and they eventually agree to some price in between. In this case it is tempting to ask the question: If Roxy and Yolande had continued their 'negotiation', what would happen to the times that they each called out?

Yolande commences with 10 o'clock, which Roxy follows up with 11 o'clock, but notice that the conclusion of the negotiation is not 10:30, midway between these initial offers. No, 10:30 is simply Yolande's first counter-offer and the conclusion of round one! The rule here, after each person has stated their opening offer, seems to be that each person 'splits the difference', going for the midpoint between the previous two times. Another way to describe this is that they halve the difference between the previous two times, and when it is Yolande's turn she *subtracts* this from the previous time, whereas when it is Roxy's turn she *adds it on*. So, measuring time in hours from 10:00, the times called out are the partial sums of the sequence $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$; in other words the sequence, $1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \ldots$, where the nth term is calculated by summing the first n terms. The sequence $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$ is a geometric sequence with initial value 1 and common ratio $-\frac{1}{2}$, which converges to $\dfrac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$. So in our example, this means a time $\frac{2}{3}$ of the way from 10:00 to 11:00, which is 10:40 pm.

There are several questions that we might pose at this point. For example, if Yolande knows that Roxy is predictable in her responses, what time should she begin with if she wants the *conclusion* to be 10:30 pm? Or 10:00 pm? To arrive at 10:30, 10 minutes earlier, Yolande just needs to begin her negotiating at a time that is 10 minutes earlier too, so the sequence now goes: 9:50, 10:50, 10:20, 10:35, … converging to 10:30, which is $\frac{2}{3}$ of the way from 9:50 to 10:50. Of course, this does assume that Roxy's first response will be to add an hour, which might not be the case if Yolande begins off the hour.

Similarly, Roxy might consider what first response to Yolande's 10:00 will converge to a more desirable conclusion for her of, say, 11:00, rather than 10:40. In this case, 11:00 needs to be $\frac{2}{3}$ of the way from 10:00 to Roxy's opening response, so Roxy needs to respond with 11:30, so that the sequence goes: 10:00, 11:30, 10:45, … converging to 11:00.

It is also interesting to consider what happens if Roxy opens negotiations and Yolande responds, with the same two original starting times. This time we have: 11:00, 10:00, 10:30, 10:15, … now converging to 10:20 (which we can think of as $\frac{2}{3}$ of the way *backwards* from 11:00 to 10:00). So perhaps (counterintuitively?) Roxy is wise to let Yolande go first (Note 1)!

Another way to express what is going on here is by means of a recurrence relation:

$$a_n = \frac{1}{2}\left(a_{n-1} + a_{n-2}\right), \text{ for } n > 2,$$

with starting values a_1 and a_2, which is a Fibonacci kind of sequence, where each term after the second is the mean of the previous two terms. It is easy to experiment with such sequences in a spreadsheet by entering two numbers into cells A1 and A2 and then entering the formula =average(A1,A2) into cell A3 and copying that cell down the column (Note 2). To prove that as $n \to \infty$ this sequence converges to $\frac{1}{3}(a_1 + 2a_2)$, as we saw above, we can find a closed-form expression for the nth term by taking $a_n = r^n$ as a trial solution, where $r \neq 0$. Substituting this gives

$$r^n = \tfrac{1}{2}\left(r^{n-1} + r^{n-2}\right)$$

and dividing through by $\frac{1}{2}r^{n-2}$ (which cannot be zero) gives $2r^2 = r + 1$, the *characteristic equation*, which has solutions $r = -\frac{1}{2}$ or 1. This means that the general solution is $a_n = A + B\left(-\frac{1}{2}\right)^n$, where A and B are constants. Substituting for $n = 1$ and $n = 2$ and solving the resulting simultaneous equations gives us

$$a_n = \tfrac{1}{3}\left(a_1 + 2a_2\right) + \tfrac{4}{3}\left(a_2 - a_1\right)\left(-\tfrac{1}{2}\right)^n, \text{ for } n > 0,$$

which we can see does indeed converge to $\frac{1}{3}(a_1 + 2a_2)$ as $n \to \infty$, since $\left(-\frac{1}{2}\right)^n$ is a null sequence.

It may be interesting for students to experiment with similar sequences using spreadsheets and to form and test various conjectures (Note 2).

Notes

1. In negotiations, people often advise that it is strategic to let the other person make the first offer. But there are also advantages in going first, in terms of establishing a psychological 'anchor' that influences the final outcome. And of course both sides can't wait for the other to go first!
2. If you wish, you can format the cells to 'Time', but I find it easier to stick to decimals in base 10.

Reference

Lodge, D. 1991 *Paradise News*, Vintage Books, London.

Keywords: Investigation; Partial sums; Real life; Recurrence relations; Sequences; Series.

Author Colin Foster, School of Education, University of Nottingham, Jubilee Campus, Wollaton Road, Nottingham NG8 1BB.
e-mail: **c@foster77.co.uk**
website: **www.foster77.co.uk**