Independent Events and Inequalities

By **Colin Foster**

Consider the probability identity below that connects the probability of *either* of two events A and B occurring with the probability of them *both* occurring:

$$P(A \cup B) \equiv P(A) + P(B) - P(A \cap B).$$

We can visualise this in terms of a Venn diagram, in which both regions representing $P(A)$ and $P(B)$ include the *intersection* region that represents $P(A \cap B)$ (shown in red in Fig. 1). In order to obtain the quantity $P(A \cup B)$, if we just add $P(A)$ and $P(B)$, we *double count* this red region (see Note). And so we have to subtract one lot of $P(A \cap B)$ to get the correct answer.

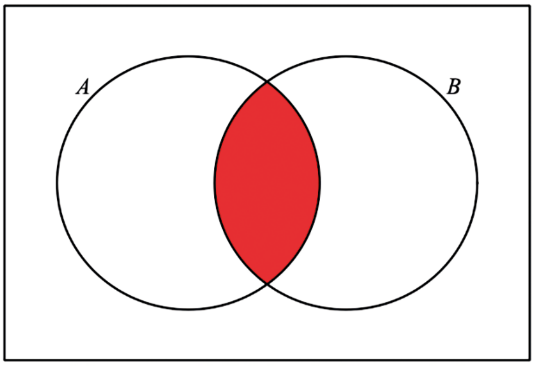

Figure 1: $P(A \cup B) \equiv P(A) + P(B) - P(A \cap B)$ visualised

Now let's suppose that A and B happen to be *independent* events.

Then, $P(A \cap B) = P(A).\,P(B)$.

Substituting this into the identity, we obtain

$$P(A \cup B) \equiv P(A) + P(B) - P(A).\,P(B).$$

Since $P(A \cup B)$ is a probability, it must lie between 0 and 1, but is this guaranteed to happen if we choose *any* values we wish (between 0 and 1) for $P(A)$ and $P(B)$? If we choose large values for them, is the product term $P(A).\,P(B)$ always guaranteed to be large enough to bring the sum of $P(A)$ and $P(B)$ down to less than 1? I don't find this obvious.

As an example, we could try, say,

$$P(A) = P(B) = \frac{9}{10}.$$

Now, $P(A) + P(B) = \frac{18}{10}$, which is a lot greater than 1. However, the product

$$P(A).\,P(B) = \frac{9}{10} \times \frac{9}{10} = \frac{81}{100},$$

and this turns out to be *just* enough to bring $\frac{18}{10}$ slightly under 1, since $\frac{18}{10} - \frac{81}{100} = \frac{99}{100}$, which is *just* less than 1. Was this lucky, or will this always happen?

Let's write $P(A) = p$, so the expression of interest is $2p - p^2$. If we complete the square,

$$2p - p^2 = 1 - (1 - p)^2,$$

and note that $(1 - p)^2 \geq 0$, we have $2p - p^2 \leq 1$, so we are always going to be OK if $P(A) = P(B)$ (Foster, 2022a).

What if $P(A)$ and $P(B)$ are different, but both (of course, since they are probabilities) lie in the interval $[0, 1]$? Let's write $P(A) = p$, $P(B) = q$, so the expression of interest is now $p + q - pq$. This expression is now a function of *two* variables, rather than one. Can we still see that it must always turn out to be less than 1?

Since $0 < p < 1$, it follows that $0 < 1 - p < 1$.

Similarly, $0 < 1 - q < 1$.

Multiplying, we obtain

$$0 < (1 - p)(1 - q) < 1.$$

Expanding,

$$0 < 1 - p - q + pq < 1,$$

and rearranging,

$$p + q - pq < 1 < 1 + p + q - pq,$$

which means that

$$0 < p + q - pq < 1.$$

Now we can use the fact that $pq < p$ (as $q < 1$) to note that $p + q - pq > q$, and (by symmetry), $p + q - pq > p$. So we have the double inequality

$$\max(p, q) < p + q - pq < 1.$$

This satisfies us that pq will always be sufficient to bring $p + q - pq$ below 1.

If we want to, we can also note that, since $pq > 0$, $p + q - pq < p + q$, giving

$$\max(p, q) < p + q - pq < p + q,$$

a tighter inequality in the case where $p + q < 1$.

This question seems like a nice opportunity to do a bit of algebra with inequalities. In an ideal world, students would worry about things like this and want to resolve them. When given a formula for probability, we ought to be concerned about whether that formula might ever give us impossible values outside the [0, 1] interval, and checking things like this seems like a very mathematical thing to encourage and take the time to do.

Note

Unless the events A and B do not overlap (are mutually exclusive), in which case $P(A \cap B) = 0$.

References

Foster, C. 2022a (May 26) 'Are two cars better than one?' [Blog post]. **https://blog.foster77.co.uk/2022/05/are-two-cars-better-than-one.html**

Foster, C. 2022b 'A clear account of events.', Teach Secondary, 11 (6), p. 13. **www.foster77.co.uk/Foster,%20Teach%20Secondary,%20A%20clear%20account%20of%20events.pdf**

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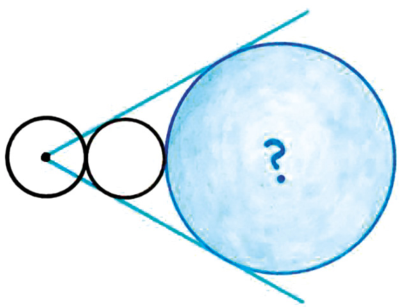
Catriona Agg's Geometry Problem 26: Solution

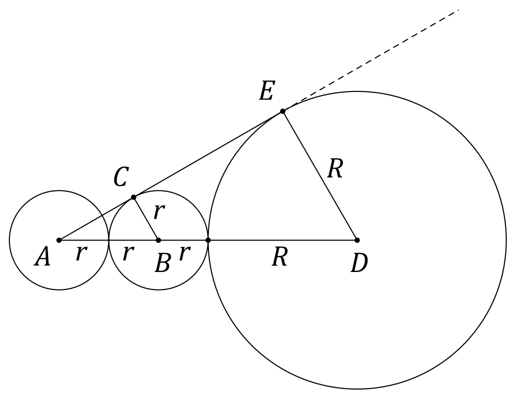
by **Chris Pritchard**

The May issue (Vol. 54, no. 3) carried the twenty-sixth of Catriona Agg's original geometry problems and asked for solutions. We were given the following diagram and asked for the area of the circle, if both of the smaller circles have unit area.



The answer is 9 (square units) and correct solutions were submitted by Jenny Burdett, David Cundy, Simon Dakeyne, Allan Duncan, Nicholas Fowkes, Mary Harris, Richard Harvey, Patricia King, Gerry Leversha, Martin Lofthouse, Gordon McConnell, Sylvia Neumann, Chris Ormell, Stuart Pattullo, Wil Ransome, Geoff Strickland and Peter Swindale.

This is a rather routine problem that is easily cracked using similar triangles. Let the radii of the smaller circles be r and that of the larger circle be R.



Then, comparing triangles ABC and ADE, we see that

$$\frac{R}{3r + R} = \frac{1}{2},$$

from which $R = 3r$ follows directly.

The area of the larger circle is $3^2 = 9$.

Special mention should be made of Gordon McConnell who provided three different solutions, including the one given here.

Finally, I must apologise for omitting Richard Harvey's name from the list of successful solvers of Problem 25 given in the September issue.

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