Colin Foster

Expression Polygons

Solving linear equations, an important mathematical technique at the core of the Algebra 1 curriculum, forms an essential foundation for more advanced work. Too often, developing fluency with linear equations entails plowing through pages of repetitive exercises. How can students master this topic while using their mathematical sense-making faculties?

In addition to generating essential practice with the technique of solving linear equations, this lesson on expression polygons engages students in authentic mathematical thinking (Foster 2013a, 2014). In particular, as advocated by the Common Core's Standards for Mathematical Practice (CCSSI 2010), this activity asks students to—

- make sense of problems and persevere in solving them (SMP 1, p. 6); and
- reason abstractly and quantitatively (SMP 2, p. 6).

The expression polygon task is a rich task that students of all ages from grade 6 upward can access.

PRACTICE GIVES OPPORTUNITY FOR FORMATIVE ASSESSMENT

The diagram in **figure 1**—an *expression polygon*—shows four algebraic expressions with lines connecting each pair of expressions. Each line forms an equation from the two expressions that it joins, and the students' initial task is to solve the six equations, writing the solution to each equation next to each line. For example, the line at the top joining $x + 5$ to $2x + 2$ corresponds to the equation $x + 5 = 2x + 2$; the solution is $x = 3$, so students write 3 next to this line. In addition to recording their solutions on the sheet, students could write out full solutions on separate paper.

Solving the equations in the expression polygon serves as formative assessment:

- Do students make errors where there are negative signs?
- Do students sometimes perform opposite operations to both sides of the equation (e.g., adding 3 to one side and subtracting 3 from the other side) rather than doing the same operation?
- Do students sometimes try to change an expression incorrectly—for example, do they change $3x$ to x by subtracting 3 instead of dividing by 3?
- Do students sometimes stop when they have found the value of a multiple of x, such as $3x$, and fail to complete the solution by dividing by the multiplier, 3?

Useful questions to ask are "What operation are you going to do to both sides of your equation?" and "Why did you choose that operation?"

If students select operations that are unhelpful, let them explore what happens when they carry out those operations. For example, with $x - 2 = 5$, if students suggest subtracting 2 from both sides, allow them to do so. Obtaining $x - 4 = 3$, they will realize that they are no closer to the solution and that

Edited by Patrick Harless

Activities for Students appears five times each year in *Mathematics Teacher*, often providing reproducible activity sheets that teachers can adapt for use in their own classroom. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more information, visit http://www.nctm.org/mtcalls.

Department editors
Sam Shah, *samjshah@gmail.com, Packer Collegiate Institute, Brooklyn, NY; and* ***David Custer,*** *dcuster@csdecatur.net, Decatur City High School, Decatur, GA*

adding 2 would have been a better move. Although poor strategic choices, such steps preserve equality, which is something to recognize and build on (Foster 2012). If students take other approaches to solving equations, such as "moving" letters and numbers from one side to the other, explore with them why these rules work and what their limitations might be. For example, to understand rules such as "change the side; change the sign," consider what identical operation is being done simultaneously to both sides.

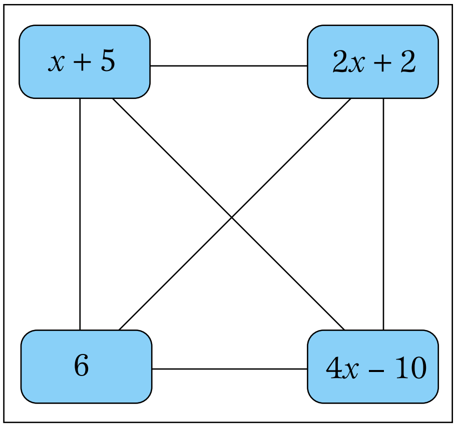


Fig. 1 Four expressions can be used to create six equations.

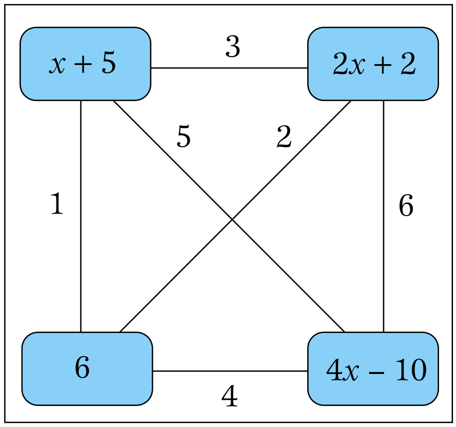


Fig. 2 The completed expression polygon shows solutions to the six equations.

FORMAT ALLOWS FOR CREATIVITY

When solving the expression polygon, students find that the solutions are 1, 2, 3, 4, 5, and 6 (see **fig. 2**). The pattern is provocative, and students typically comment on it (Foster 2012). Uncovering an interesting pattern leads naturally to a challenge: "Can you make up an expression polygon of your own that has a nice, neat set of solutions?"

Of course, students decide what counts as "nice" and "neat." They might choose to aim for the first six even numbers, first six prime numbers, first six squares, or some other significant set of six numbers. Regardless of the specific targets students choose, the trial and improvement involved in producing their expression polygon will generate useful practice in solving linear equations. Because students are aiming for particular solutions, they must deconstruct the equation-solving process. As students gain confidence in solving equations, they can focus their attention on their strategic decisions about which expressions to choose. In this way, the task naturally differentiates for students.

Producing a nice set of solutions is more difficult than it first appears—try it! Initially, aiming for integer solutions may be enough of a challenge. Help students who are struggling by encouraging them to think about how they could simplify the task to get started. For example, they could begin with an expression triangle, with three expressions, rather than an expression square. (**Fig. 3** shows an expression triangle created by a grade 8 student.) Another possibility is to remove the two diagonal lines from the expression square, so that only four equations are required. After solving these simpler versions of the problem, students will be better prepared to attempt the original task. Although at different levels of complexity, all versions of the task require strategic choices and problem-solving skills while also developing fluency with linear equations.

Two additional approaches have helped students who struggle to make progress with the main activity:

1. *Begin with the original expression polygon* (see **fig. 1**). Try starting with the original expression square and changing one or two of the numbers in one or two of the expressions. What happens when you solve the new equations that you get? Why?
2. *Model an experimentation process.* Choose two small positive integers. If the student offers, for example, 3 and 5, write down $3x + 5$ in one of the empty expression boxes. Then ask the student: "Can you put a number into one of the other expression boxes to create an equation with an integer answer?" If the student is still unsure, encourage him or her to begin with any number, solve the resulting equation, and then adjust the choice if the solution is not an integer. By trial and improvement, he or she will begin to discover what is needed to obtain integer solutions.

Adapting the original expression square can be a productive way to generate new sets of solutions. For example, one grade 7 student carefully doubled each expression, with the intention of obtaining the solutions 2, 4, 6, 8, 10, and 12, and was surprised when he found that he obtained exactly the same solutions as for the original expression polygon. Through discussion with other students, he realized that this happened because multiplying both sides of an equation by the same factor leaves the solution unchanged. Older students realize sooner that solutions are preserved when scaling or

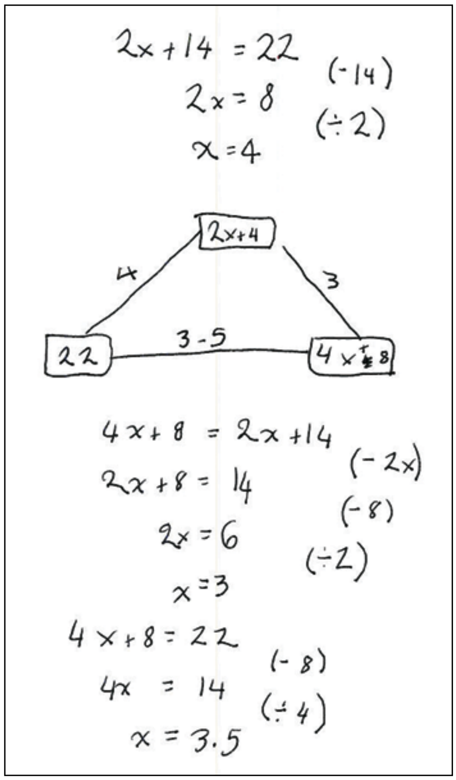


Fig. 3 An expression triangle was created by a student in grade 8.

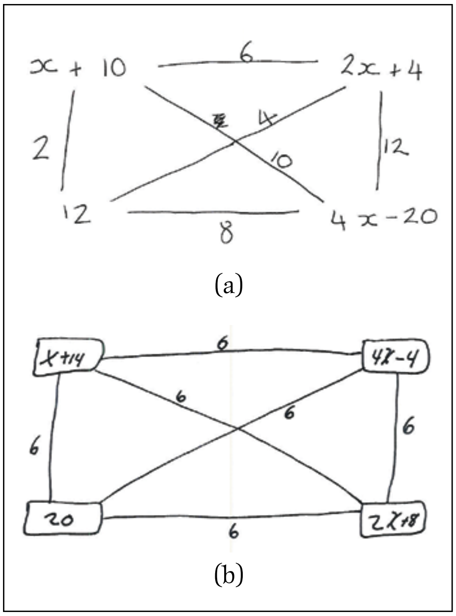


Fig. 4 Students may wish to create expression polygons with particular solutions such as all even integers (**a**) or all equal integers (**b**).

translating all the expressions by the same constant.

Adopting a more sophisticated approach, another student replaced every x with $(1/2)x$ (so that $4x - 20$ became $2x - 20$, etc.) and then doubled all the expressions (to clear the fractions), obtaining the expression polygon shown in **figure 4a**, whose solutions are the first six even numbers. Another "neat" set of solutions that students sometimes consider involves making all the solutions the same (see **fig. 4b**).

Students sometimes observe patterns among the polygons themselves. For example, expression triangles have three solutions, but expression squares have six solutions, rather than four. In general, an expression n-gon will have $(1/2)n(n - 1)$ solutions, one per diagonal. Since the number of required solutions is quadratic in n, completing an expression n-gon becomes rapidly more difficult as n increases. A grade 9 student produced the expression pentagon shown in **figure 5**, in which all the solutions are integers, albeit not distinct. Students may consider creating an expression polyhedron, in three dimensions, and realize that, for example, an expression square is equivalent to an expression tetrahedron and an expression cube is equivalent to an expression octagon. For further challenge, students could include quadratic expressions (Foster 2013b) or ask whether a given six-tuple can give the solutions to an expression polygon. For example, is it possible to design an expression square with the solutions 1, 1, 1, 1, 1, and 2?

DISCUSSION REINFORCES STUDENT OBSERVATIONS

Toward the end of the lesson, ask students to share the expression polygons that they have created and to talk about how they made them. Some helpful prompts include these:

- "How did you choose your expressions?"
- "What changes did you make to your expressions and why?"
- "Which was the hardest part of making your expression polygon and why?"

Often students succeed with three expressions but have difficulty adding a fourth. They may need to change one of the three expressions that "work" to find a way forward. Describing their process and how they worked through their difficulties can reinforce the value of persistence.

Students may have noticed that the set of solutions is preserved when scaling or translating all the expressions by the same constant, and this observation could be valuable to draw out. They may have noticed that it is relatively easy to generate an expression polygon in which all the solutions are the same; if so, it could be useful to ask them to explain a method for doing this. Students may comment on the fact that an expression n-gon has $(1/2)n(n - 1)$ solutions; if not, then it may be worth asking them about why an expression triangle has three solutions but an expression square has six solutions, rather than four.

The expression polygon task provides extensive practice in solving linear equations while developing students' creativity and problem solving. Beyond fluency with equation-solving techniques, the activity moves students' attention to a higher strategic level, thereby deepening their understanding of algebraic operations and equality.

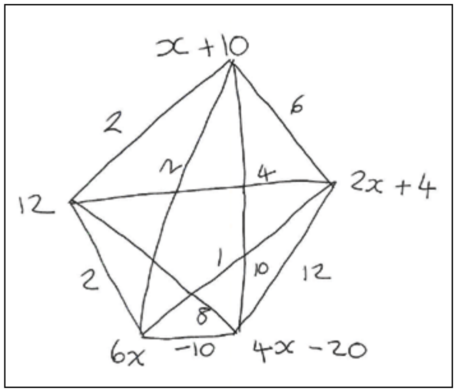


Fig. 5 A completed expression pentagon shows solutions to ten equations.

REFERENCES

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Foster, Colin. 2012. "Connected Expressions." *Mathematics in School* 41(5): 32–33.

———. 2013a. "Mathematical Études: Embedding Opportunities for Developing Procedural Fluency within Rich Mathematical Contexts." *International Journal of Mathematical Education in Science and Technology* 44(5): 765–74.

———. 2013b. "Connected Quadratics." *Teach Secondary* 2(1): 46–48.

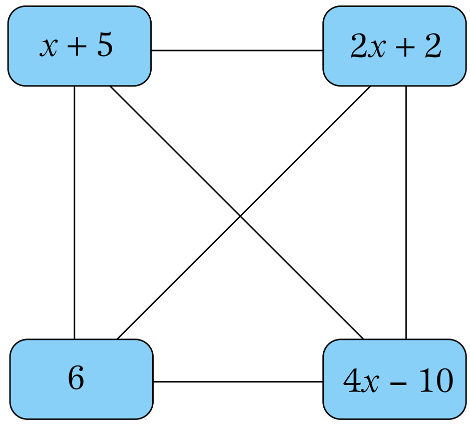
———. 2014. "Mathematical Fluency without Drill and Practice." *Mathematics Teaching* 240: 5–7.

COLIN FOSTER, colin.foster@nottingham.ac.uk, is an assistant professor at the Centre for Research in Mathematics Education in the School of Education at the University of Nottingham in England. His research interests are in the design of rich, open mathematical tasks and how teachers and learners use, adapt, and own them in the classroom.

more4U

For a customizable activity sheet, go to **www.nctm.org/mt**.

ACTIVITY: EXPRESSION POLYGONS



The figure at right is an *expression polygon*. Each pair of algebraic expressions is joined by a line segment. Each line segment represents an equation equating the expressions that it connects. For example, the line segment at the top represents the equation $x + 5 = 2x + 2$.

1. Write the six equations represented by the six line segments in this expression square.
2. Solve your six equations.
3. Describe any patterns you notice among your six solutions.
4. Construct another expression polygon containing different expressions. Can you make the solutions to your expression polygon a "nice" set of numbers?

The remaining two extension questions might prompt further discussion:

5. How could you make an expression polygon in which all the solutions were 7?
6. How many solutions will an expression 10-gon have? What about an expression n-gon? Why?

THINGS TO CONSIDER

Number of Expressions

An expression triangle would be easier to begin with than an expression square. An expression pentagon would be challenging. An expression cube, in three dimensions, could be very challenging!

Type of Solutions

Here are some possible challenges:

- Make all the solutions integers.
- Make all the solutions distinct (i.e., different) integers.
- Make the solutions a "nice" set of numbers, such as consecutive even numbers, or square numbers, or prime numbers.

Is it always possible to design an expression polygon that will produce any given set of numbers? Why or why not?

Type of Expressions

It is best to start with linear expressions of the form $ax + b$, where a and b are constants. Including one or more quadratic expressions of the form $ax^2 + bx + c$, where a, b, and c are constants, could give some equations with two solutions, so two numbers might be needed on some line segments.

From the August 2015 issue of MATHEMATICS teacher