Colin Foster suggests that atomisation is not the only way to limit the demands on learners' working memory.

Alternatives to atomisation

Breaking down mathematical processes for learners into sequences of small steps—atomising them—is often presented to teachers as a necessity. Anyone who questions this is likely to be accused of suffering from the 'curse of knowledge'. Teachers are already familiar with the mathematical process, so we underestimate how difficult it might be for a learner to get to grips with it. We overestimate how much a novice learner can handle at any one time without their working memory becoming catastrophically overloaded. The answer is to atomise, so that the learner can focus on mastering each step individually, and then build those separate units back up into the complete process. When presented like this, atomisation may seem essential to scaffolding learning. What would be the alternative – to try to do everything all at once? A recipe for disaster, surely!

I am not saying that atomising processes is always a bad idea. But I want to argue that there are alternative ways to scaffold processes that retain a more holistic and connected perspective, without overwhelming working memory. These approaches seek to make a complicated, multistep process manageable without sacrificing a big-picture understanding of what is going on.

As an example, let's take the classic process of adding two fractions, where the larger denominator is not a multiple of the smaller one, such as $\frac{3}{4}+\frac{2}{5}$. In cases like this, *both* denominators will need to be converted into a new, common denominator before the addition can take place.

Figure 1. Retaining arbitrary knowledge consisting of someone's preferred list of steps for lots of different processes across lots of different topics requires constant opportunities for retrieval practice. As more processes are learned, this is like spinning plates—unless you return sufficiently often to each plate, it will come crashing down.

The atomisation approach would involve listing all of the steps and sub-steps necessary to complete this process and teaching each one separately before combining them. I have no doubt that this can be effective, in the sense of enabling a majority of students to successfully perform the process for themselves after a reasonable amount of teaching and practice. But it can be very challenging to remember the details of all of these separate steps once the class has moved on to a new topic. I find that even teachers who themselves are perfectly competent at adding fractions struggle to remember a few topics later exactly what steps they taught to their class: "What are the six steps for adding fractions? ... Oh, hang on, sorry, I mean what are the seven steps for adding fractions?" Small steps can often be broken down into yet smaller steps, so there is not always an obvious endpoint, and different teachers will come up with different steps. The teacher can add fractions perfectly well, but cannot remember the particular atomisation that they taught the class. This suggests to me that although the desired process involves performing these steps, experts do not do it by remembering the separate steps. Teaching by using these steps, when we do not do it this way ourselves, means being very confident that it is definitely optimal for novice learners, and I am not sure that it necessarily always is.

Remembering arbitrary knowledge versus building connected knowledge

Retaining arbitrary knowledge (Hewitt, 1999) like this, consisting of someone's preferred list of steps for lots of different processes across lots of different topics, is likely to be very challenging for the learners too. If these details are not to be forgotten, constant opportunities for retrieval practice will be needed, which becomes an increasing problem as more processes are learned. I think of this as being like spinning plates—unless you return sufficiently often to each plate, it will come crashing down (Figure 1).

Although long-term memory is effectively infinite, and people can in theory store an unlimited number of processes, the learner will only be able to retrieve those stored memories if they practise retrieving each of them frequently enough. Any plate spinner, however good, will eventually reach their limit and be unable to keep any more plates simultaneously in the air. Due to the shape of the Ebbinghaus forgetting curve (Murre & Dros, 2015), plates that were set up longer ago, and have been returned to more times, will last longer before they need another twist. Even so, they will have to be returned to eventually, to keep that memory accessible. This means that the more mathematics that you learn in this kind of way, the more of these plates you have to keep spinning in the air, so the more difficult the subject is likely to feel. Eventually, we might worry that the time left for learning anything new will be squeezed out by all the retrieval practice required for all the many things that have been previously learned up to that point!

Alternatively, with a more connected approach to learning, the more mathematics you learn the *easier* the subject gets. Rather than having an increasing number of plates to keep

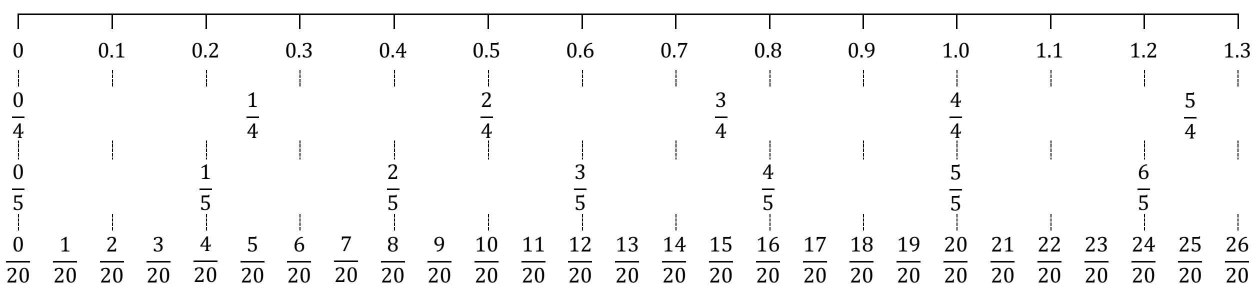

Figure 2. A fraction number line.

spinning, you have an increasing number of conceptual connections between different things, meaning that ideas make more and more sense as you go on, and you have less and less to consciously remember. Many learners who succeed in mathematics say that one reason that they like it is that they do not have to remember much. Might this perspective be something that all learners could experience?

Doing things differently

How else, besides breaking the process down into separate steps, might we scaffold a process such as adding two fractions? To keep working memory manageable, we want to avoid learners being expected to immediately do every aspect of the process all at once. But we also want to avoid a dull, step-by-step treatment that leaves learners with a long shopping list of things to remember to do, and that will inevitably fade over time.

One approach to scaffolding through connections, rather than atoms, would be to present learners with a number line / fraction wall diagram such as the one shown in Figure 2 (A PDF version of this is available at *foster77.co.uk/Fraction%20number%20line.pdf*). Ideally, I would print this landscape on A3 paper, so that there is plenty of space to annotate, and share one sheet between two students, to encourage discussion.

This is a busy diagram with a lot going on, so learners will need to be given some time to make sense of it. I find that learners do not mind lots of things on the page as long as they are not being rushed through it. You could initially ask them to 'Say what they see'. Depending on what comes out of this, you could follow up on essentials with:

What patterns do you see? ...vertically? ... horizontally?

Why are there some gaps?

How do the decimals at the top fit in?

This discussion will lead to learners talking about equivalent fractions being in the columns and families of fractions with the same denominator being in the rows, and ordered from left to right. The decimals at the top of the diagram are tenths, and a row for tenths (written as fractions) could be added if desired. Every other quarter has no corresponding tenth, because tenths do not go into odd quarters. The number line deliberately extends a bit beyond one in order to discourage learners from thinking that fractions are 'numbers less than one'. That also gives us more possibilities for adding fractions without spilling outside the bounds of the number line.

After discussing all of this, I would start by asking learners to work out something like 'seven twentieths plus five twentieths', perhaps working with this in words before writing it as $\frac{7}{20} + \frac{5}{20}$. Staying with words initially would delay any notational issues and capitalise on the natural scaffolding of the language, where 'seven somethings plus five somethings' really sounds like it ought to be 'twelve somethings'. Adding the 7 and the 5 is natural—everyone

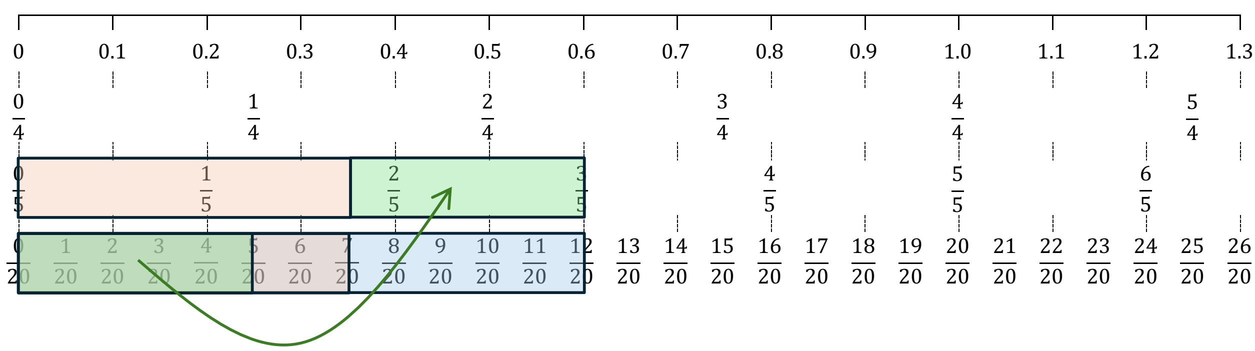

Figure 3. Bar models overlaid onto the number line.

will do that. If learners also add the 20s as well, then for me the problem here is not that they are failing to follow a rule ("Don't add the denominators!"), but rather that they are misunderstanding what the '20' means. There are not 20 of anything, so it makes no sense to add the 20s together. Twentieths are the units that we are counting in, just as 7 cm + 5 cm is equal to 12 cm, not just '12'. If no one adds the two 20s, I would deliberately suggest this: "Oh, did you forget to add the 20s as well?" and get learners to explain why it would make no sense to add the 20s. If learners are 'with' what is going on, they will tell you that it would be ridiculous to add the 20s.

Someone might notice that the answer, $\frac{12}{20}$, could be simplified to $\frac{3}{5}$, perhaps by finding it above $\frac{12}{20}$ on the diagram. (Someone might also suggest $\frac{6}{10}$, perhaps prompted by the 0.6 at the top of the diagram.) If no one mentions that, I would ask if there is a simpler way to give the answer. The idea that the fraction nearest to the top of the diagram will be in the lowest terms will emerge from this.

I would also take time to decide whether the answer $\frac{12}{20}$ is sensible or not. The addends $\frac{7}{20}$ and $\frac{5}{20}$ are both less than $\frac{1}{2}$ (How do we know this?), and so their sum must be less than 1 (Why?). Learners might be able to sharpen up this estimate.

We can shade in the bars of the bar model if that helps, overlaid on top of the diagram. For me, a number line is never very far from being a bar model (Figure 3).

So far, we might not have written anything down. As far as the learners are concerned (and possibly an observer), we have not 'done any work' yet—we are just talking about things. In the atomisation-oriented classroom next door, learners would by now have answered dozens of questions, probably on mini-whiteboards, practising multiples and equivalent fractions, as pre-requisite knowledge checks. I am not saying that that would be a bad lesson, but the focus here is quite different, and more on the ideas behind what is going on. The rationale here is that investing this time in making sense will save us lots of problems later and avoid us having to invent (and practise) numerous rules to prevent learners from going off track.

Next I would do $\frac{7}{20} - \frac{5}{20}$. (I would always choose to begin by adding a larger fraction to a smaller one, so that I can just change the plus to a minus and ask what will happen now.) I have never encountered any child who will not by this point say $\frac{2}{20}$. There is nothing to subtracting fractions that we have not already completely set up – it doesn't need to be explicitly taught. I also think that if we have done the adding properly then it would be quite surprising if anyone did $\frac{7-5}{20-20} = \frac{2}{0}$. But if they do then we can have the discussion again about what the denominator means. Again, the point if this happens is not that we forgot to tell them not to do this, but that someone doing this is showing us that they do not understand what the denominator means (i.e. 'the namer'). We also might want to end up considering the issue that division by zero is undefined.

Some learners may be happy to reduce $\frac{2}{20}$ to $\frac{1}{10}$ or 0.1. The diagram continues to offer lots of support with this, because it has all of the numbers in the correct positions. Some learners will say that $\frac{2}{20}$ must be equal to 0.1 because it appears directly beneath it in the diagram, even if they do not yet see why $\frac{2}{20} = \frac{1}{10}$. This means that if they are shaky on equivalent fractions they can still join in everything, and they are getting the idea of adding and subtracting, even if not the simplifying yet. I think that this way of working is potentially just as accessible as the small steps are in the atomisation classroom next door, but the pieces of the jigsaw here are of a different nature. Our small pieces are not individual, linear steps in the finished process, but elements of understanding the entire process.

Next, we might try $\frac{7}{20} + \frac{1}{4}$. That looks tricky—learners might think impossible—because are we supposed to be counting in 20ths or in 4ths? (Early lessons on fractions may benefit from speaking of 'fourths' instead of 'quarters'.) Eventually, someone will notice that $\frac{1}{4} = \frac{5}{20}$, so the sum becomes $\frac{7}{20} + \frac{5}{20}$, and turns out to be exactly the same calculation as the one that we have just done! Learners can be asked to find many possibilities of 20ths and 4ths, or 20ths and 5ths, that they can add together, and write them out. Before long they will be confident suggesting things like $\frac{3}{4} + \frac{2}{5}$, taking each fraction down into the 20ths row, and then coming back up at the end (if possible) to simplify the final answer. There is lots that you can do staying with just the fractions on this diagram and using 20ths as the common denominator. I would not be in a hurry to move on from this particular set of examples until I was sure that everyone was very confident.

Those moving faster could be asked to find examples of $\frac{\square}{\square} + \frac{\square}{\square} + \frac{\square}{\square}$ or $\frac{\square}{\square} + \frac{\square}{\square} - \frac{\square}{\square}$, perhaps aiming for a specified final answer. Part of the scaffolding in this setup is knowing that 20ths are going to be the common denominator. Within this context, there is lots of practice converting where the numbers are relatively easy, on the assumption that learners are more likely to be comfortable with multiples of 4 and 5. This could easily take a whole lesson, with students writing out lots of correct additions and subtractions, which are easy for their partner or the teacher to quickly check.

In the next lesson we might recreate the diagram, rather than being given it. Can the students remember what was on it? Once we agree what the different rows were and how far they went horizontally, can they fill in the details? After recapping calculations like $\frac{3}{4} \pm \frac{2}{5}$, with that diagram, it could be time to make a different diagram, perhaps with 12ths or 16ths. (Why would including 13ths or 14ths not be very useful?) Different learners could create different diagrams and then share their calculations for others to check. ("Make up four examples, where three are correct and you smuggle in one incorrect one. See if your partner can find the incorrect one.") Eventually, with all of this specific experience, it becomes the job of the learners— not the teacher —to explain how to generalise. So, the teacher asks the learners, "How do we find a suitable common denominator? How do we convert the starting fractions into ones that have that denominator?" No one has to memorise the answers to these questions or write them down as 'rules', because this is just describing what we have all been doing.

At the end of all this, there is nothing much to remember besides the terminology – language like 'numerator', 'denominator' and 'common multiple'. And we have used these words so much throughout that they have become familiar through use. The process has been scaffolded, but without atomising it into individual, spoon-fed steps. Whether this is better than atomising is for the reader to consider – and perhaps try out. But I think you can argue that its demands on working memory do not necessarily have to be any greater.

References

Hewitt, D. (1999). Arbitrary and necessary part 1: A way of viewing the mathematics curriculum, *For the Learning of Mathematics,* 19(3), 2–9.

Murre, J. M., & Dros, J. (2015). Replication and analysis of Ebbinghaus' forgetting curve, *PloS one*, 10(7), e0120644. *https://doi.org/10.1371/journal.pone.0120644*

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