# Is there any place for rote learning in mathematics?

Colin Foster questions whether rote learning is necessarily bad in every situation.

The phrase 'rote learning' is sure to set alarm bells ringing for many readers of *MT*, as indeed it does for me. By 'rote learning', or 'learning by heart', I mean deliberately repeating something until it is remembered, through force of repetition. This is typically contrasted with 'learning through understanding', in which learners think about *why* something is true and try to connect new knowledge to other things that they know. Rote learning appears to treat learners as empty vessels needing to be filled, and repeating word-for-word what the teacher says feels like an attempt to exercise power and control over learners' emerging thinking. Through this, learners perhaps learn not to trust themselves but to rely on whatever they are told – and there goes critical thinking and any hope of developing citizens able to advocate for what they believe is right in a democratic society.

But I am unsure whether this is quite the right way to think about rote learning. Perhaps, when I criticise rote learning, I am being a bit hypocritical (Foster, 2019), because I notice that I tend to believe that things are correct if I have heard them lots of times. Of course, this is not an infallible rule, and generally it is a highly unreliable way to arrive at the truth. For instance, we frequently hear negative stereotypes, and that does not make them true. But I do think that the familiarity of 'what sounds right' is a big part of what is going on in my head when I am doing mathematics.

For example, if you were to ask me, "What is five times five?", I would instantly respond with "25". If you followed up by asking me, "How do you know that?", I could of course give all kinds of explanations, such as:

$$5 \times 5 = 5 \times \frac{10}{2} = \frac{5 \times 10}{2} = \frac{50}{2} = 25.$$

But, of course, this is not how I actually 'did it'. Sometimes, in classrooms, the teacher will ask learners, "How did you do it?" And the learner knows that what is expected is a series of steps, like those set out above, even though, if they answered quickly, it is extremely unlikely that they really did any of that. The honest answer might be more like, "I just knew it". That would be my honest answer in this case, so does that suggest that I am reliant on rote memory for this? In other words, I have just heard "5 × 5 = 25" so many times that it feels right.

You might take the view that automaticity is a good thing, but that being automatic doesn't imply that 25 was, somewhere in my distant past, rote-learned. Perhaps as a child I had some rich, connected experiences of seeing 5 × 5 = 25 in ways that did not depend on rote repetition. I do not now remember, but that may well be true. And, indeed, I can of course outline a series of steps like those above – a sort of proof, if you like. But I do not need to go through that every time I wish to draw on the fact that 5 × 5 = 25. I just know it, and I think that that is really just because I have seen it that way so many times.

It is problematic to demand more than this. Even if I insist on explaining 5 × 5 = 25 step by step, as above, any one of those steps itself depends on various other necessary facts. We can always ask another, "But how do you know *that*?" kind of question at each point. For example, "How do you know that $5 = \frac{10}{2}$? Or how do you know that you can interchange the order of multiplication and division in the $5 \times \frac{10}{2} = \frac{5 \times 10}{2}$ step? Or how do you know that $\frac{50}{2} = 25$?" We can also ask more meta questions, like, "How did you decide that it would be a good idea to replace 5 by $\frac{10}{2}$ rather than, say, $\frac{15}{3}$?" We can always go on and on asking "Why?" about things that came naturally to the person. And maybe they can answer those questions too, but none of this is really at the bottom of what is going on when they said "25".

I have seen learners try to justify their knowledge of something like 5 × 5 = 25 by recourse to algorithms like short multiplication: "I would do it like this", they say, writing:

$$\begin{array}{r} 5 \\ \underline{\times\ 5} \\ \underline{25} \end{array}$$

and narrating this by saying "five ... times ... five ... makes ... twenty-five".

Or, using a rectangular model:

| × | 0 | 5 |
|---|---|---|
| 0 | 0 | 0 |
| 5 | 0 | 25 |

where they might narrate, "Zero times zero is zero, five times zero is zero, zero times five is zero ... and five times five is ... twenty-five". Clearly, neither of these procedures is a tool to obtain the answer. They are representations that depend on already knowing the answer.

It seems to me that ultimately I know that 'five fives are twenty-five' because it sounds right through familiarity, and I see that as a consequence of 'rote' (if you like) repetition over a long time. It is almost like completing a line from a poem that is lodged firmly in my head, like knowing that "The owl and the pussycat..." is followed by "...went to sea...". 'Five fives are twenty-five' comes along with things like 'six sixes are thirty-six'. Indeed, there is an interesting pattern here, because, if you extend this to 7 × 7, you get, 'Seven sevens are (twenty, thirty) **forty** (five, six) **seven**'. Except, of course, that 7 × 7 = 49, not 47. Based on rhythms and patterns, 'seven sevens are forty-seven' sounds much better. So, it is not 'sounds right' in the sense of fitting a pleasing pattern that counts. To me, and to anyone who knows their tables, 'seven sevens are forty-nine' is the one that sounds right, and 'seven sevens are forty-seven' sounds wrong. The familiarity of the right answer has more weight than the pattern of 5, 6, 7 that leads to the wrong answer. Indeed, I cannot think of any very quick way of verifying that 7 × 7 = 49, for instance, or even seeing why it must be less than 50 rather than greater than 50. Like 5 × 5, it is familiar to me, I suspect, solely through repetition.

There is an obvious parallel here with language. One aspect of the privilege of being a first-language user of English is that the only rule of grammar you need to know is: 'If it sounds right, then it probably is'. For instance, I personally cannot explain why adjectives in English have to be in the order 'opinion-size-age-shape-colour-origin-material-purpose' (Forsyth 2014, p. 40), nor can I even recall that list without looking it up. But, like every other first-language English speaker, I 'just know' that:

> you can have a lovely little old rectangular green French silver whittling knife. But if you mess with that word order in the slightest you'll sound like a maniac. It's an odd thing that every English speaker uses that list, but almost none of us could write it out. (Forsyth, 2014, p. 40)

As Tim Dowling (2016) has put it, "You simply can't say My Greek Fat Big Wedding".

I suspect that, when doing mathematics, we all rely more than we might realise on what feels right at the time. For example, in simplifying $2x - x$, which feels more right as an answer: $x$ (the right answer) or 2 ('two $x$ and we take away the $x$')? Does cancelling the $x$'s in $\frac{x+y}{x}$ look right or wrong? It feels natural to me to replace $x(x + 1)$ with $x^2 + x$, but a situation such as $\sqrt{x+1}$ or $x(x + 1)^2$ would feel completely different, and might trigger different actions. In contrast, a learner at various stages of developing their algebra might feel that all of these are of the same kind, and it is surely that awareness, or feeling of what's right, that we want to educate. Although I can explain these things if pressed, thoughts of why are not uppermost when I am proceeding through some algebra in the service of some larger problem that I am interested in solving.

The phrase rote learning perhaps summons up an image of a room full of children chanting "five fives are twenty-five". But perhaps there is also a kind of rote learning that happens more naturally, through repeated exposure over a longer period of time, in a less pressured way. I have encountered a lot of situations over the years in which 5 × 5 was taken to be 25, and very few situations in which it was taken to be anything else. Cumulatively, that leaves me as someone who feels right saying 25 and wrong saying anything else. Is this rote learning? This is still remembering through the force of repetition, rather than anything deeper, so is perhaps still fairly described as rote. A mathematics educator once told me that they so often refer to 7 × 8 = 54 as a common misconception that they have confused themselves about whether the correct answer is 54 or 56. This suggests that the force of repetition can indeed be a powerful thing, even for an expert mathematician with vast amounts of conceptual understanding.

So, I do not think I feel any hatred for repetition of things like 7 × 7 = 49. Where I become uncomfortable with rote learning is when the things being repeated are somewhat less than absolute truths. Especially when they might be regarded as examination-passing hacks, invented by the teacher and contingent on knowing the kinds of questions typically asked. And especially if, on closer inspection, they are not really right, or will expire as the students move on to the next stage of learning mathematics (see Dougherty, Bush and Karp, 2017). For example, I have heard about teachers asking, "What do we do when we see a right-angled triangle?" and the learners are supposed to respond, "Pythagoras or trigonometry". And yet why should we do anything? The right-angled triangle might be part of a problem concerning symmetry or tessellation or ratio or angles or bearings or circle theorems or area. In each case, there are other things that might be 'done' than these two.

To take another example, the teacher might ask, "What does 'of' mean in maths?" And the learners are drilled to reply "times". But that is not always the case. Multiplication may be relevant in lots of situations, such as 'a fraction of', but when we read $f(x)$ as '$f$ of $x$' or $\sin x$ as 'sine of $x$', there is certainly no 'times' hiding inside the 'of', although it might look like it symbolically. Similarly, the 'of's in '3 out of 5 is 60%' and 'Five ducks were sitting by a pond and three of them swam off. How many were left?' have nothing to do with multiplication and should definitely not cue that operation.

The same thing arises when a teacher asks the question, "What do brackets mean in maths?", expecting the knee-jerk answer, 'Do that first'. When evaluating 4(3 + 2), we could indeed work out the 3 + 2 first, and then multiply by 4. But we could also work out the 4 × 3 and the 4 × 2 first and then add them up. So, it feels misleading to say that brackets tell us what to do first. And brackets in circumstances such as $\sin(x+1)$, or with coordinates (3, 2), or vectors $\binom{3}{2}$, or matrices $\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}$, or sets {2, 3}, have nothing to do with doing things first. Unfortunately for simplistic rote learning, brackets, and the word 'of', do not mean one thing in mathematics. It depends on the context.

So, I think that rote learning is definitely bad when the things being learned by rote are not 100% correct. But, when they are, then I think I have no particular issue with repeated reciting of important and fundamental number facts, or identities like $\sin^2 x + \cos^2 x \equiv 1$. I do not think this has to be oppressive, because I want students to get to the point where what feels right is useful to them, and I think that that largely happens through repeated exposure. I think it is absolutely fine to just know things like this through repeated exposure.

Ideally, sometimes rote recall can be achieved in ways that avoid mindless repetition. One way to eventually just know the outcome of a calculation, for example, might be to start by repeatedly calculating it, until you no longer need to do so. If, to start with, you keep working out 5 × 5 by halving 5 × 10, then eventually you will start to say, "I think it's 25", and you will do half of 5 × 10 merely to check. And then, beyond that, you will remember it as a fact in its own right, and just know it, and the feeling that you need to check it will fade away. This is learning through repetition and familiarity, and, this way, you will always be able to go back to working it out, if needed, if you have forgotten. However, with more complicated facts (for example, 7 × 7), demanding that learners work it out from scratch every time they forget it is perhaps less productive, and might feel punishing. In cases like this, or with learning reasonably large prime numbers, mindless repetition might, I think, be defendable.

Overall, I think there is no need to apologise for just knowing things like 5 × 5 = 25. If necessary, we can go back and construct an argument or proof to justify things like this. But we cannot be doing this all the time, to endlessly reassure the teacher that we are sense-making, or else we could never get on and access bigger mathematical problems.

## References


Dougherty, B. J., Bush, S. B., and Karp, K. S. (2017). Circumventing high school rules that expire. *The Mathematics Teacher, 111*(2), 134-139.

Dowling, T. (2016, September 13). Order force: the old grammar rule we all obey without realising. *The Guardian*. https://www.theguardian.com/commentisfree/2016/sep/13/sentence-order-adjectives-rule-elements-of-eloquence-dictionary?CMP=share_btn_tw

Forsyth, M. (2014). *The elements of eloquence: Secrets of the perfect turn of phrase*. Berkley Books.

Foster, C. (2019). Doing it with understanding. *Mathematics Teaching, 267*, 8–10.

Foster, C. (2020). Trusting in patterns. *Mathematics in School, 49*(3), 17–19.



**Colin Foster is a Reader in the Department of Mathematics Education at Loughborough University.**

c@foster77.co.uk

https://www.foster77.co.uk/

https://blog.foster77.co.uk/