ABOLISH DEGREES!

By **Colin Foster**

No, this isn't an article about shutting down universities and preventing anyone from graduating! This is about a unit of angle measurement that I increasingly think is more trouble than it's worth. It doesn't seem to me that degrees bring enough benefits to outweigh their drawbacks and, in an ideal world, I don't think they would earn their place in the school curriculum. It's probably unrealistic to try to get rid of degrees – certainly not anytime soon – but, in my ideal mathematical world...

Often, progressions across school mathematics aim to take students from less general, less elegant, less powerful ideas to more general, more elegant, more powerful ideas. But, occasionally, things seem to go in the reverse direction, and the 'inevitable march of progress' seems strangely interrupted. And angle measurement seems to me to be one of those situations.

Understanding angles can be difficult (Alyam, 2022), but, from an early age, children naturally begin talking about angles in terms of 'turning' or 'twisting'; for example, when rotating door handles or opening and closing anything that has a hinge, or just when moving their own bodies around in space. Turning all the way round on the spot is one full turn; turning half way round, to face the opposite direction from the way you were originally facing, is half a turn. Quarter turns of the minute hand on an analogue clock are important for telling the time. Describing angles as fractions of a turn is a natural and, I think, highly mathematical thing to do.

But, later on in school, students are introduced to *degrees* as a unit of angle, and I think this is really a kind of *backwards* step in their journey. Unlike fractions of a whole turn, degrees are an *arbitrary* unit (Hewitt, 1999). The question "Why do we have 360° in a full turn?" can only be answered in terms of the *history* of mathematics, not in terms of *mathematics* itself. It's (possibly) a convenient choice, but it isn't *necessary* (Hewitt, 1999) – it could have been otherwise, and indeed there exist other choices (Note 1). I think introducing degrees sets learners back on their mathematical journey, and makes learning about angles harder than it needs to be, because it's with degrees that they then go on to carry out angle calculations involving polygons, and angles associated with parallel lines – and also their first steps in trigonometry. There is more extra work needed than we sometimes appreciate for learners to become familiar with 90° as a right angle, and 270° as three-quarters of a turn. Scaling up all our fractions of a turn by 360 is extra unnecessary trouble, and, apart from providing an opportunity for some multiplication practice, seems of no value. Our own 'curse of knowledge' may make us feel that this is no big deal, but having 90 for a quarter turn (rather than a rounder number like, say, 100) leads to some large and awkward numbers. When it comes to something like length, having an arbitrary unit of measurement, such as a metre, is unavoidable, but it is entirely unnecessary for angle, since we have fractions of a turn.

If learners go on beyond age 16 to study more mathematics, at that point they encounter the radian, which, after all those years working with degrees, they may well perceive as an ugly and difficult unit (a full turn is now 6.283185... radians – where is the beauty in that?). If radians are introduced before calculus, they tend to be justified as convenient because formulae for the arc length and area of a sector of a circle of radius r take on seemingly 'nicer' forms if θ, the angle at the centre, is measured in radians than if it is measured in degrees:

formula	*with θ in degrees*	*with θ in radians*
length of arc	$\frac{2\pi r\theta}{360}$ (Note 2)	$r\theta$
area of sector	$\frac{\pi r^2\theta}{360}$	$\frac{1}{2}r^2\theta$

With radians, the formulae have smaller numbers and look neater, and the factors of π are eliminated. But this benefit may seem slight to students, and it isn't really why we use radians, so I think they are bound to be a bit underwhelmed with radians if this is how we justify their importance. The real value of radians comes when we encounter calculus and things like Taylor series, where radians are the only sensible, 'natural' unit to use. Compare what happens with radians and degrees:

formula	*with x in radians*	*with x in degrees*
Taylor series for x	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$	$\frac{\pi x}{180}-\frac{(\pi x)^3}{3!\,180^3}+\frac{(\pi x)^5}{5!\,180^5}-\frac{(\pi x)^7}{7!\,180^7}+\cdots$
Derivative of $\sin x$	$\cos x$	$\frac{\pi}{180}\cos x$
Derivative of $\sin^2 x$	$2\sin x\cos x$	$\frac{\pi}{90}\sin x\cos x$

The interesting thing for me about this progression from angle as a fraction of a turn, through angles in degrees, to angles in radians is that radians are really extremely similar to the 'fraction of a turn' measure that the students began with. There is a kind of circularity here (pun intended), and the digression into degrees interrupts the story. Radians are really no more 'a unit' than 'fraction of a turn' is, and it is a bit odd to treat them as though they are. If learners write the word 'radians' after, say, '$\theta=2\pi$', or use the superscript C notation, as $\theta = 2\pi^C$, this can be quite misleading, as radians are dimensionless ratios of lengths (i.e., $\theta = \frac{\text{arc length}}{\text{radius}}$), and so no 'units' are needed when an angle is given 'in radians' (see Wheeler, 1958). I find that students sometimes become confused about this, and write things like '$\pi = 180$', while simultaneously knowing that $\pi \approx 3.14 < 180$. Converting 90° into $\frac{\pi}{2}$ radians is not really like converting 90 inches into 228.6 cm.

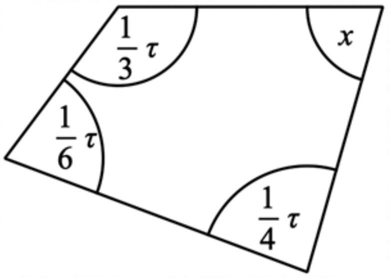
This brings me to the main difficulty with radians, which is that having '*half* π for a *quarter* turn" is quite annoying. With pi day on the horizon (**https://www.piday.org/**), I have been revisiting the arguments for tau ($\tau = 2\pi$) in preference to π (see **https://tauday.com/** and Bartholomew, 2014), which really boil down to the observation that 2π just comes up a lot more frequently than π, and so shouldn't 2π ($= \tau$) be our more natural unit? Although we can write $c = \pi d$ to connect the circumference and diameter of a circle, we are generally more interested in radii than in diameters, and so, perhaps, $c = 2\pi r$ is more relevant. And therefore wouldn't it be better to absorb the '2' in the constant of proportionality and write $c = \tau r$, with $\tau = 2\pi$, so that π relates to a semicircle but τ relates to the much more important situation of a whole circle? Those in favour of tau argue that '2π' comes up in important formulae far more often than 'π' does alone, so we should jettison π in favour of τ. This would certainly seem to apply in our discussion, where 2π, not π, represents the full turn of a whole circle.

However good an idea tau might be, school mathematics is perhaps not the place for individual schools or teachers to try to begin this revolution. But it does seem that we suffer with radians by our choice of π over τ, and this is entirely a problem of our own making, because we already have the very handy unit of 'a full turn', which even young children are quite used to. The table below shows how the different measures are related, and the similarity between the first and last columns is quite striking. Effectively, 'tau', the Greek letter 'T', can be taken as standing for 'turn': "a quarter turn is an angle of a quarter tau". What could be simpler?

fraction of turn	*angle in degrees*	*angle in radians*	
		in terms of π	*in terms of* τ
0	0°	0	0
$\frac{1}{4}$	90°	$\frac{\pi}{2}$	$\frac{\tau}{4}$
$\frac{1}{2}$	180°	π	$\frac{\tau}{2}$
$\frac{3}{4}$	270°	$\frac{3\pi}{2}$	$\frac{3\tau}{4}$
1	360°	2π	τ

Like radians, 'fraction of a turn' is a natural unit, and I think could be the clearest and easiest way for school students to think about angle. There is really no way in which a radian is any 'cleverer' than this sense which children acquire from quite an early age of angle as a fraction of a whole turn. So, "an angle of a half" should not be taken to be $\frac{1}{2}$°, or even $\frac{1}{2}$ radian = $\frac{180}{2\pi}$, but simply as 'half a turn', or $\frac{\tau}{2}$.

Degrees seem to have little utility in school mathematics, as far as I can see, beyond an opportunity for numeracy practice. We could do angle calculations using fractions of a turn, as in Figure 1. Sectors of pie charts would be easy to calculate, based on percentages (Figure 2), and drawn with 100% angle measurers (see **www.tarquingroup.com/tarquin-pie-chart-scales-pack-of-10-flexible-percentage-circles.html**), rather than 180° or 360° ones. A 100% angle measurer can be thought of as being 'in radians', with tau as the 'unit' (e.g., 20% is 0.2τ).



$$\frac{1}{4}+\frac{1}{6}+\frac{1}{3}=\frac{3}{4}$$
$$1-\frac{3}{4}=\frac{1}{4}$$
$$x=\frac{1}{4}\tau$$

Figure 1. Calculating angles in fractions of a turn (taus). (Diagram not drawn accurately.)

Colour	*Frequency*	*Percentage*	*Angle*
Red	14	28%	0.28τ
Yellow	15	30%	0.30τ
Pink	11	22%	0.22τ
Green	10	20%	0.20τ
Total	*50*	*100%*	τ

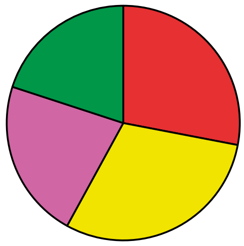


Figure 2. Calculating percentages in fractions of a turn (taus).

I suppose someone might say that at least degrees have familiarity going for them. Everyone knows them, since they met them in school – although this is a circular argument, since I'm questioning what they *should* meet in school. But I wonder how familiar degrees really are. I recently had some flooring put down at home, and I asked the tiler if he could rotate the design he was doing "by 90 degrees". If there's anyone who is likely to 'use angles' in their everyday life, it's surely someone who fits floor tiles for a living, and this tiler was experienced (and did an excellent job). He replied, "How much is that? I was never any good at maths at school." It turned out that "a quarter of the way round" was perfectly clear.

So, in conclusion, I think my preferences for angle measures are:

$$\text{radians (with } \tau \text{ better than } \pi) \approx \text{fraction of a turn} \gg \text{degrees}$$

Shall we ditch the degrees?

Notes

1. For example, the mysterious, almost-never-used, 'gradians' button within the 'Deg Rad Grad' function on old calculators. With this unit, 100 grad = a right angle, so there are 400 grad in a full turn.
2. It may be tempting to cancel this down to $\frac{\pi r\theta}{180}$, but this loses the transparency of the process as $\frac{\theta}{360}\times 2\pi r$; i.e., a dimensionless fraction of the whole circumference, $2\pi r$.

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