Are Words Sometimes Better Than Formulae?

By **Colin Foster**

If, 50 years ago, you had asked a pupil in the UK what Pythagoras' Theorem was, they might have answered: "The square on the hypotenuse is equal to the sum of the squares on the other two sides". Today, if you asked a pupil that same question, I think they would almost certainly say, "a squared plus b squared equals c squared". It's interesting to consider the advantages and disadvantages of each of these responses. The first response is verbal; the second is symbolic. And, in the history of mathematics, people wrote formulae in words for a long time. Have we lost something by having pupils become 'too symbolic, too soon'?

Perhaps the verbal version of Pythagoras' Theorem suggests a more 'conceptual' sense of what is going on, but I'm not sure about that. It's still stating a 'fact', and is a form of words that generations of people have probably memorised with little understanding. The use of the word 'hypotenuse' centres mathematical terminology and tells us that we're dealing with right-angled triangles, whereas the symbolic version is devoid of context; the 'c^2', for instance, could invoke completely different scenarios, such as $E=mc^2$, as picked up on in the cartoon shown in Figure 1.

Figure 1: Pythagoras vs Einstein Funny Math Science T-shirt (www.teeshirtpalace.com)

For both versions of Pythagoras' Theorem, we have to supply the necessary triangle ourselves, but for the symbolic version it's critical that we label the sides appropriately (Figure 2), which means that the verbal version might perhaps be easier to apply when faced with a non-standard labelling (e.g., Figure 3). To apply the symbolic version, $a^2 + b^2 = c^2$, to Figure 3 requires either relabelling the triangle or saying, "Let their a be my c, ..." and so on, which consumes working memory and can easily lead to errors.

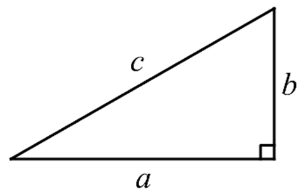

Figure 2. $a^2 + b^2 = c^2$

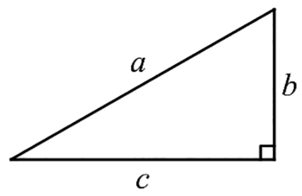

Figure 3. $a^2 = b^2 + c^2$

In contrast to the case with Pythagoras' Theorem, there are other situations in which pupils today may be much more likely to respond verbally than symbolically. For example, they would say "Half the base times the height" for the area of a triangle, rather than "bh over two" (Note 1), and something like "The angle at the centre is twice the angle at the circumference" for circle theorems. Perhaps the first of these happens because the formula for the area of a triangle is likely to be encountered before students are comfortable with 'symbolic algebra', but that would not seem to be the case for the circle theorem. Whatever the reason, could it be that, in those cases where the symbolic version (as appears on formulae sheets) seems to dominate, there might be something to be gained by encouraging verbal versions?

Perhaps it seems unnecessary to set verbal and symbolic as alternatives, against each other like this. Presumably we want students to be comfortable with both. But I think there is perhaps a 'first response' which is primary, which comes to mind first, and which is the first thing a student goes to. And I don't think we should assume that that should necessarily have to be the symbolic version. It sometimes seems to be taken for granted that expressing a mathematical relationship in words is a kind of early, tentative step, which we hope will be supplanted later on by the more robust and general 'symbolic' version. But I think that's too simplistic. As with Pythagoras' Theorem, words can be just as general as symbols, and sometimes words may be more convenient, and perhaps more easily make the structure apparent. Of course, there are limits – I wouldn't fancy trying to make a verbal version of the quadratic formula! – but sometimes the benefits of the verbal seem worth considering.

I've collected in Figure 4 some examples of formulae which I often teach in words – not as an alternative to symbols, of course, but as the intended principal go-to. I wonder if I am unusual in doing this? Written down on the page, the word version is always going to look longer and harder – some of them seem almost like a poem – and part of the beauty of symbols is their economy of expression. But, even if they are longer, I'm not sure that *remembering* the word versions is necessarily harder. Words sometimes seem to illuminate the structure more clearly, and they can be very handy when adapting to awkwardly-labelled situations (e.g., Figure 5). In these cases, words can feel like an orthogonal channel from the algebraic letters, that avoids the letters in the standard formula becoming jumbled up with the ones in the specific example it's being applied to. But, clearly, words have their limits when it comes to precision. The priority of operations is not nailed down precisely in the cosine rule in Figure 4, with "minus twice the product of those sides, multiplied by the cosine of the angle between them". And "that same integral" is clunky and potentially ambiguous in the integration by parts formula.

	Words	Symbols
Pythagoras' Theorem	The square on the hypotenuse is equal to the sum of the squares on the other two sides.	$a^2 + b^2 = c^2$
Area of a triangle	Half the product of the adjacent sides, multiplied by the sine of the angle between them.	$\frac{1}{2} ab \sin C$
Division of fractions	To divide by a fraction, multiply by its reciprocal.	$\div \frac{a}{b}$ is equivalent to $\times \frac{b}{a}$, where $a, b \neq 0$
Cosine rule	The opposite side, squared, is equal to the sum of the squares of the other two sides, minus twice the product of those sides, multiplied by the cosine of the angle between them.	$c^2 = a^2 + b^2 - 2ab \cos C$
Product rule	The first factor stands and watches, while the second factor gets differentiated. Then the second factor stands and watches, while the first factor gets differentiated. And then you add them up.	$(uv)' = uv' + u'v$
Quotient rule	The denominator is: the bottom squared. The numerator is: bottom times top differentiated, minus top times bottom differentiated.	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, $v \neq 0$
Integration by parts	The first, times the integral of the second. Minus the integral of the derivative of the first, times that same integral.	$\int uv' \, dx = uv - \int u'v \, dx$

Figure 4. Words versus symbols for some common formulae

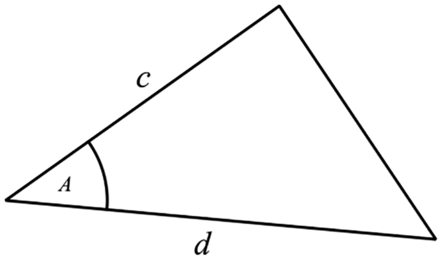

Figure 5. Find the area of this triangle using the formula '$\frac{1}{2}$ab sin C'.

I'm always keen to try to avoid pupils mindlessly substituting into formulae. I have seen students in science lessons desperately searching through their sometimes extensive formula sheets for any formula that might contain letters that match the starting letters of some of the key words in the question they are trying to answer ("It's about 'pressure' – which formulae have a '*p*' in them?"). Perhaps this sort of thing would be less of a problem for students who have a mental databank of *verbal* theorems?

Looking at Figure 4 also makes me realise that there are many choices with the *symbolic* versions that I think are rarely discussed. For example, is the product rule easier to conceive of (and/or use) as $uv' + u'v$, with both products having the factors in the same order, or as $uv' + vu'$, with the "don't differentiate – do differentiate" order consistent instead? Is the cosine rule better thought of in the form

$$a^2 + b^2 = c^2 + 2ab\cos C,$$

as a 'correction' to Pythagoras' Theorem for non-right-angled triangles, or as

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

where it is all set up for calculating c^2, and thus c? Of course, we want students to be able to cope with *all* these situations, and be able to operate flexibly, rearranging equations when needed (see Foster, 2021). But I still think there is a question to ask about which version might be preferable as the *primary* one, from which others are then derived.

One issue is how helpful the different versions (verbal/symbolic) might be conceptually, and another (perhaps sometimes conflicting?) issue is how easy the different versions might be to compute or operate with. In the old days, with old-fashioned 'immediate-execution' calculators, pupils might often inadvertently calculate $(\pi r)^2$, when they meant to calculate πr^2. Teachers would sometimes try to avoid this error by teaching the formula for the area of a disc as "r times r times π", and write it as $r \times r \times \pi$, so abandoning the usual algebraic conventions that would lead to πr^2. Was this warranted, or just an examination-passing 'hack'?

And what about informal, 'abbreviated formulae', such as '$\sin^2 + \cos^2 \equiv 1$' (read as "sine squared plus cos squared equals one"), where there is no variable, and ("cos of A plus B equals cos cos minus sine sine")? Do these have a place or are they too dangerous to allow?

Note

1. Expressing this in words can sometimes reveal that students are confused about whether this formula is $\left(\frac{1}{2}b\right) \times h$ or $\frac{1}{2}(b \times h)$ or $\frac{bh}{2}$, not realising that these are all equivalent (see Foster, 2010).

References

Foster, C. 2010 *Resources for teaching mathematics 14–16*. Continuum.

Foster, C. 2021 'On hating formula triangles', *Mathematics in School* 50 (1), pp. 31–32.

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Author: Colin Foster, Department of Mathematics Education, Schofield Building, Loughborough University, Loughborough LE11 3TU.

e-mail: **c@foster77.co.uk**

website: **www.foster77.co.uk**

blog: **blog.foster77.co.uk/**