Armchair Responses

by **Colin Foster**

Sometimes attempting to address or resolve one error or misconception seems simultaneously to lead to a new one. The more aware I have become of this phenomenon, the more instances I seem to see of it in the classroom. I wonder if this could be an inevitable feature of learning – perhaps particularly in mathematics? Maybe this is what progress looks like: acquiring more and more sophisticated misconceptions?

The lesson that got me thinking about this was an introductory one on set theory. A student had a question similar to this:

A = {2, 3}, B = {1, 4}.
What is A ∪ B?

The student had written {2, 3, 1, 4} and the teacher was explaining that this was wrong "because you've got the numbers in the wrong order".

But a set is an *unordered* collection, so how can the order be "wrong"? I felt uneasy about the teacher's response to what the student had written. "We have to write them in order" is a convention – a convenient one, certainly, if you want see clearly what you've got, or compare two sets, but not fundamental to the idea of a set. If a set contained the number 2, a triangle and the city of Birmingham, it is not obvious that there is a natural order in which to write these. So it seemed to me that addressing the student's non-conventional ordering in this way was perhaps in danger of confusing them about what a set is.

Following some armchair reflection, I think I would wish to have responded with another question – "What is B ∪ A?" – to see whether the student would write {1, 4, 2, 3}. Since sets are unordered collections, {2, 3, 1, 4} = {1, 4, 2, 3}, and the teacher could then ask what *else* is equal to these, and what other possible ways of writing this set the student could think of. Of the 24 possible ways (Note 1) of writing this one set, {1, 2, 3, 4} is obviously a convenient one to use as the standard.

Later, in the same lesson, a situation like this appeared:

A = {2, 3}, B = {1, 3}.
What is A ∪ B?

A student had written {1, 2, 3, 3}, this time taking care to write the elements in numerical order. The teacher pointed out that this answer was wrong for a different reason: "You've got two 3s in there and there should only be one". When presented with the correct answer {1, 2, 3}, the students asked, "What about the other 3? Where did it go? Why do we miss it out?" and the response was that there is a rule that "we only write each number once".

Again, I felt that the teacher's response to the student's error was in danger of inadvertently creating or reinforcing a different misconception, in this case the idea that {1, 2, 3, 3} is different from {1, 2, 3}.

Again, following my armchair reflection, I think I would wish to have addressed this by asking how many elements are in the set {1, 2, 3, 3}? If the student answered 3, the correct answer, the teacher could just ask them why they've written one of those elements (but not the others) twice. But if, as is more likely, they answered 4, then you could ask them to tell you what these four elements are:

Student "1"
Teacher "OK" [writes it on the board]
Student "2"
Teacher "OK" [writes it on the board]
Student "3"
Teacher "OK" [writes it on the board]
Student "3"
Teacher "I've already got that".

At this point, the student is likely to say something like, "But there's another one", and the teacher can say, "What do you mean another one?" Thinking of numbers as unique elements, like people, is quite helpful here. For example,

Members of football club F = {Faridah, Iman, Moses}

Members of drama club D = {Leillah, Moses, Qayla}

People who are members of either or both of these clubs, F ∪ D = {Faridah, Iman, Leillah, Moses, Qayla}

We wouldn't say 'Moses' twice, just because he's in both clubs, as he's still just one person. There are only 5 people in F ∪ D, not 6.

However, the more I think about my alternative response here, the more I realize that my people analogy also has the potential to create or reinforce further misconceptions. Although this wouldn't arise until a more advanced stage, treating sets as physical containers of physical objects, while very natural when beginning in set theory, can also lead to problems later on. For example, a set such as {A, {A}} would be physically impossible – how can the same 'object' A be both outside *and* inside the inner container (Brown, 2012, p. 81)? – but it is perfectly acceptable mathematically.

I think there is no careful pedagogical route through mathematics that avoids all the classic misconceptions. For example, it is hard to imagine a trajectory through learning number that did not involve a 'multiplication always makes things bigger' misconception arising at some point. To feel that we have to circumvent these possible misconceptions at all times would make us too scared to say or do anything. Instead, it seems to me that these misconceptions are part of making sense of mathematics, a natural part of the process, and we should expect students to exhibit them for a time before replacing them later on with better, more sophisticated (mis)conceptions (Smith III, Disessa and Roschelle, 1994).

Armchair responding like this can be a fun game – and a useful one, I think. The question isn't, "What would you have said if you'd been the teacher?" Who knows what I might have said – maybe exactly the same as the teacher did. The question is, from the comfort of my armchair, in my ivory tower, "What do you *wish* you would have said if you'd been the teacher?" We can all improve on things from the leisurely armchair perspective. So it is very important to stress that none of this is to say that the teacher who was actually there in the classroom was doing a bad job – not at all. The point is that we all might benefit from contemplating, from time to time, how we might do things better than most of us probably do most of the time.

Apparently General George Patton said that "No good decision was ever made in a swivel chair," and of course we need to try to keep our feet on the ground. But I think we can improve what we do in the classroom by reflecting at our leisure on things that happen there. I feel sure that doing this armchair responding task – *"What would you wish to have said?"* – especially collaboratively with colleagues, whether student teachers or experienced teachers, can be a great way to help us develop our real responses when we are put on the spot in the hurly-burly of the mathematics classroom.

Note

1. This is ignoring writing the same number more than once, as I address later on.

References

Brown, J. R. 2012 *Platonism, Naturalism, and Mathematical Knowledge*, Routledge, Abingdon.

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