Choosing the Best Proofs

By **Colin Foster**

It is a truism to say that proof is fundamental to the nature of mathematics (e.g., see Brown, 2010; Hamkins, 2020). Unlike scientists, mathematicians do not merely accumulate over time more and more evidence for their claims, constantly revising and improving on tentative, provisional theories, gradually adding to their confidence as new data comes in and groping towards the truth. Mathematics develops, not so much by refining the accuracy of previous statements, but by building on theorems held to be completely true, in order to find and prove new theorems, also held to be completely true. Students sometimes find it strange that ancient theorems in mathematics have not required revision. When asked to research some history of mathematics and produce posters, one Year 8 student wrote on theirs: "Pythagoras' theorem was discovered over 2000 years ago, and is still used today." You can detect the student's surprised tone! Putting it like that does make it sound remarkable: what theories in biology from 2000 years ago are still useful today? [See Note 1.]

Unlike in university mathematics, proofs are rarely seen as central in school mathematics. Sometimes they appear in investigative project work, where students discover patterns, make conjectures and then try to justify them with arguments that, at their best, constitute (perhaps informal) mathematical proofs. But their value is in their *creation* and the mathematical reasoning involved: theorems concerning magic squares or 'frogs' (see Andrews, 2000), say, are not important results in and of themselves. In this article, I am concerned with the other appearance of proofs in school mathematics: *proofs of key theorems* (see Cirillo, 2009). My sense is that these proofs, if they appear at all in the classroom, tend to be presented by the teacher, or arise out of highly-scaffolded tasks, often near the beginning of a topic, before the theorems themselves are then used repeatedly to answer questions and solve problems.

For example, a teacher might prove the theorem that the interior angle sum of a plane triangle is 180°, and then the students' main task will be to use (rather than prove) this theorem to find missing angles in plane figures. The skills needed for doing the exercises are often very different from those needed to prove the result or understand the proof. Success with simple exercises may largely come down to "number bonds to 180", whereas the proof relied on knowledge of alternate angles on parallel lines – these are not the same thing. This means that the proving part of the lesson sequence may sometimes appear tangential, and even frivolous, in that, if some or all of the students remain bewildered by it, this is unlikely to prevent them from completing the subsequent exercises satisfactorily. Indeed, this reasoning may push the busy teacher with a crowded curriculum, and what they perceive as 'weak' or uninterested students, to omit the proof altogether.

I think this disconnect between what is required to understand the proof and what is needed for completing the subsequent exercises is very common in school mathematics. For example, success with applying something like the sine rule might depend mainly on a student's ability to label triangles correctly using the standard conventions, substitute into formulae, handle trigonometric and inverse trigonometric functions on the calculator (including getting the calculator into 'degrees' mode) and round answers, whereas proving the theorem depends very little on any of these things. Similarly, success using the quadratic formula to solve equations depends on very different skills from those needed to prove it, such as completing the square. Long teacher introductions can result if the teacher first spends time explaining the proof, and then, following that, begins explaining how to use the result to answer questions, before the students can get on and do anything [Note 2].

However, assuming that the teacher is committed to offering a proof to the class, one question that arises is: Which proof should I present? Many results in elementary mathematics have countless alternative proofs. How do we select the one we will offer? Are some proofs better, pedagogically, than others? Hamkins (2020) discusses the question of why anyone would ever need more than one proof, as, certainly, one valid proof is always sufficient to establish a claim. In a sense, all valid proofs should be equally convincing. However, he notes that "Proofs tell us not only that a mathematical statement is true, but also why it is true" (p. xiii), and "different arguments, especially when they are extremely different and highlight different fundamental aspects of a topic, deepen our mathematical understanding and appreciation of a mathematical phenomenon" (p. 9). Viewed this way, it might be tempting to say that we should expose students to *multiple* different proofs for each theorem, comparing and contrasting the insights that they provide, and perhaps then even asking them which one(s) they prefer and why [Note 3]. However, even if more than one proof is going to be shared, selection is still needed – for example, for the (perhaps extreme) case of Pythagoras' theorem, it is well known that there are literally hundreds of available proofs. So, some criteria are needed for choosing which one(s) to focus on.

Here I suggest eight possible pedagogical criteria that could influence a teacher's preference for one proof over another. I will then try to test these out for the case of Pythagoras' Theorem.

1. It's quick.

This criterion is perhaps often paramount – classroom time is always highly limited. A teacher might wish to spend as little time as possible on a proof that they suspect many of their students may not follow, and won't need for the subsequent exercises – but they want to be able to say later on that, "We proved it, don't you remember?" Something quick, even if not completely comprehensible to everyone, at least may make the point that 'mathematics involves proving things'.

2. It's easy to understand and convincing.

I think this is often in tension with #1. A longer proof, with more steps, each of which is simpler, may overall be easier to follow than an elegant one-liner that requires deep thought. But visual proofs, where available, often satisfy both #1 and #2.

3. It uses techniques similar to those needed for its subsequent application.

Although, as mentioned above, often the techniques needed for the proof are quite different from those needed to *apply* the theorem, this is not always the case. For example, circle theorem proofs often involve finding pairs of radii that form isosceles triangles, and this can also be a relevant strategy when applying these theorems to solve problems.

4. It provides an opportunity to review or see applications of recently-taught techniques / theorems.

This may be a driving factor in curriculum sequencing, where, for example, we choose to introduce theorems about angles on parallel lines *before* we introduce the theorem that the interior angle sum of a plane triangle is 180°, so that we can use alternate angles in that proof. This principle is of course fundamental in university mathematics, where theorems constantly build on previous ones. But often, in the school curriculum, a theorem appears because it is useful without there having been the necessary development of the techniques needed to prove it, e.g., formulae for the volume of 3D solids (Foster, 2015).

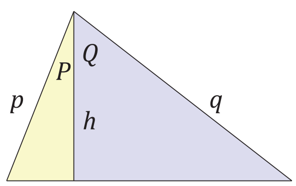
5. It teaches an important method of proof that students need to know.

Here, the *method* of the proof is the objective more than (or equal with) the theorem itself. For example, older students need to know proof by contradiction, induction, etc., and younger students might need to know the process of proof by direct argument for showing similarity and congruence of triangles. The results proved by these methods may be less important than how they were obtained. For example (see Note 4), using induction to prove that the sum of the first n fourth powers of the integers is

$$\tfrac{1}{30}n(n+1)(2n+1)(3n^2+3n-1).$$

6. It's maximally general.

Sometimes a proof addresses only a particular special case, and other cases really ought to be treated separately. Indeed, sometimes this is overlooked, wittingly or unwittingly, and the special case is taken to stand for *all* cases. For example, the identity $\sin(P+Q) \equiv \cos P \sin Q + \sin P \cos Q$ can be proved completely generally; for example, by using coordinate geometry or similar triangles. Alternatively, a quick way is to use the diagram below, consisting of two right-angled triangles, along with the $\frac{1}{2}ab\sin C$ formula.



Area of large triangle = Area of yellow triangle + Area of blue triangle

$$\tfrac{1}{2}pq\sin(P+Q) = \tfrac{1}{2}hp\sin P + \tfrac{1}{2}hq\sin Q$$

Dividing through by $\frac{1}{2}pq$,

$$\sin(P+Q) = \frac{h}{q}\sin P + \frac{h}{p}\sin Q = \cos Q\sin P + \cos P\sin Q$$

This is a nice approach, but it doesn't generalise to angles $P + Q > 180°$, even though the identity does. And the method doesn't generalise to the $\cos(P + Q)$ identity either. So, is there any benefit to using this, if you have to use a different method for proving the $\cos(P + Q)$ identity?

7. ***It's elegant and aesthetically pleasing.***

   Proofs may be opportunities for awe and wonder, such as the wow moment students may obtain from Euclid's proof of the infinity of the primes. This may be enough to justify the inclusion of a particular proof. Movshovitz-Hadar (1988, p. 34) commented that "all school theorems ... possess a built-in surprise ... by exploiting this surprise potential their learning can become an exciting experience of intellectual enterprise to the students".

8. ***It's canonical.***

   Often because of #7, some proofs are famous and part of the mathematical furniture, and every student should encounter them. For example, the classic proof by contradiction of the irrationality of $\sqrt{2}$ is part of the canon (see Kinnear & Sangwin, 2018), even if alternatives may arguably offer greater insight (Coles, 2005; Foster, 2021).

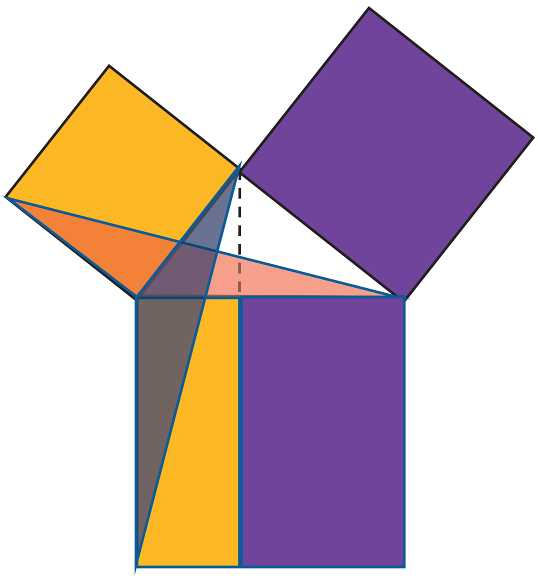
Clearly, there are many possible tensions among the criteria in this list. Do we prioritise a clever, one-off 'trick' proof, that does the job quickly, but doesn't contain ideas likely to be useful elsewhere, or do we opt for a longer, harder-to-follow proof that uses standard techniques that will work again and again? What if the former might offer a bit more of a 'wow'? The resolution of this might depend on the teacher's perceptions of the attitudes of the particular students, the overall diet of recent lessons, and many other factors.

Borovik and Gardiner (2019, p. 13) described Pythagoras' theorem as "one of the first truly surprising results in school mathematics", so in the diagrams that follow I have had a go at putting down my personal judgements on these criteria for a selection of different proofs of Pythagoras' theorem. I did not find this at all easy to do, and I would be very interested to hear what conclusions others might come to, or what different criteria might be viewed as important.

Five proofs of Pythagoras' Theorem

For more details, see Acheson, 2020, Beckman, 2020, Burk, 1996, and also my discussion in Foster, 2018.

A. Euclid



The area of the red obtuse-angled triangle is equal to half of the area of the orange square.

The blue obtuse-angled triangle is congruent to the red obtuse-angled triangle (SAS), so has the same area.

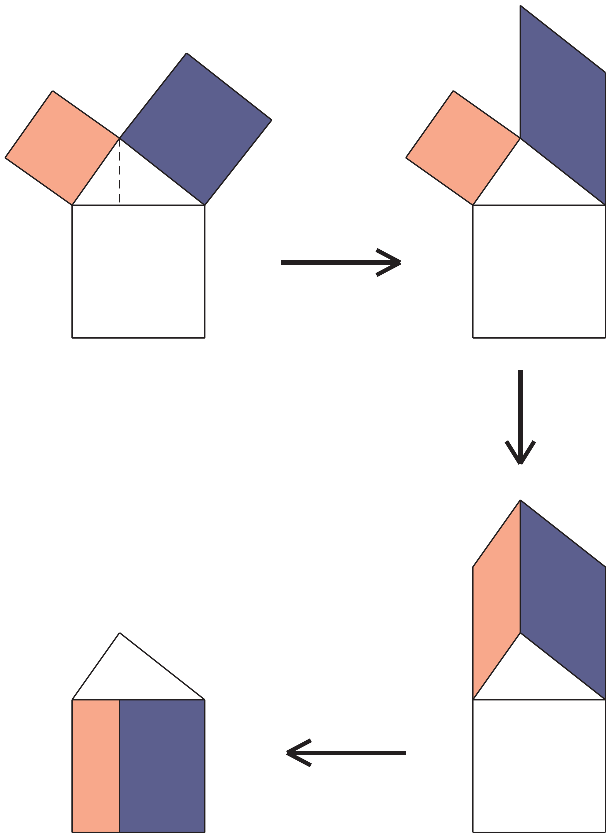
The area of the blue obtuse-angled triangle is equal to half of the area of the orange oblong.

So, the area of the orange square is equal to the area of the orange oblong.

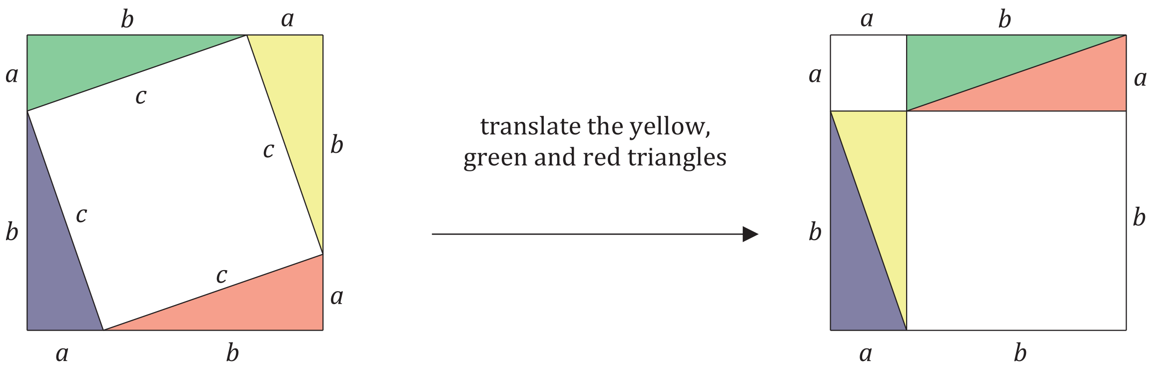
An identical argument establishes that the area of the purple square is equal to the area of the purple oblong.

So, the sum of areas of the orange square and the purple square is equal to the area of the large square at the bottom.

B. Shearing and translating

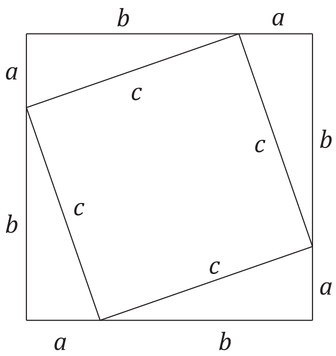


C. Dissection (example) [Note 5]



The white area in the left figure must be equal to the white area in the right figure: $c^2 = a^2 + b^2$.

D. Visual $(a + b)^2$ expansion [Note 5]



Equate the area of the large square, calculated in two different ways:

$$(a+b)^2 = 4\left(\frac{1}{2}ab\right) + c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

$$a^2 + b^2 = c^2.$$

E. Burk (1996)

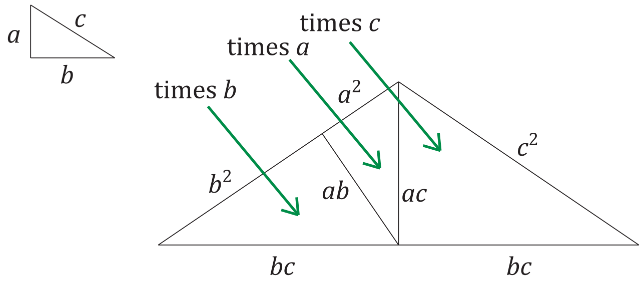
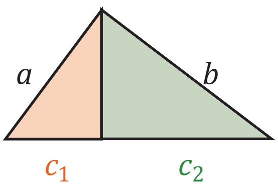


Figure reconstructed from: Burk, F. 1996 'Behold: The Pythagorean Theorem'. *The College Mathematics Journal,* 27(5), 409.

F. Similar-triangles (attributed to Einstein)



$c_1 + c_2 = c$

Split a right-angled triangle into two smaller right-angled triangles (orange and green in the figure).

If you check the angles, all **three** of these triangles are similar.

This means that

$$\frac{c_1}{a} = \frac{a}{c} \text{ and } \frac{c_2}{b} = \frac{b}{c},$$

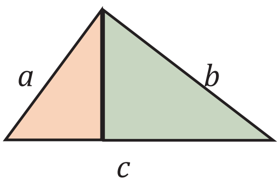
$$\text{so } c_1 c = a^2 \text{ and } c_2 c = b^2.$$

Adding these equations,

$$(c_1 + c_2)c = a^2 + b^2$$

$$c^2 = a^2 + b^2.$$

G. Area-scale-factor



This proof uses the same three, similar right-angled triangles as in proof F.

The areas of similar triangles are proportional to the squares of their corresponding sides. Taking the hypotenuses of the three triangles as the corresponding sides, we have:

Area of original triangle = area of orange triangle + area of green triangle

$$kc^2 = ka^2 + kb^2$$

where k is a non-zero constant.

So, $c^2 = a^2 + b^2$.

Acknowledgement

I presented some of the ideas in this article in my plenary at the Mathematical Association Annual Conference in April 2021.

Notes

1. It is important to acknowledge that what we call "Pythagoras' theorem" was known over a thousand years before Pythagoras was born (see Robson, 2008).
2. This would seem to contrast with situations in which we could regard the routine work that students focus on as a kind of 'direct proof', such as when teaching solving equations or finding missing angles or lengths in polygons. We could even choose to frame these tasks as 'proofs':

   Theorem: The third interior angle in any triangle with interior angles 30° and 60° is 90°.

   Proof: The interior angle sum of a triangle is 180°, so the third angle is $180 - 30 - 60 = 90°$.
3. Hamkins (2020) does this beautifully in his book, in particular, providing seven different proofs that $n^2 - n$ is even. You might like to try to anticipate the different proofs before looking at Chapter 2 of the book.
4. See *Faulhaber's formula* for details of $\sum_{k=1}^{n} k^p$ for other p.
5. These figures are taken from Foster (2018).

Proof	1. quick	2. easy to understand	3. uses techniques similar to those needed to apply it	4. reviews or applies recent content	5. teaches an important method of proof	6. maximally general	7. elegant and aesthetically pleasing	8. canonical
A. Euclid				■	■	■	■	■
B. Shearing and translation		■		■	■	■	■	
C. Dissection (various)	■	■		■		■	■	
D. Visual $(a+b)^2$ expansion				■		■	■	■
E. Burk		■		■		■	■	
F. Similar-triangles (Einstein)				■	■	■		
G. Area-scale-factor	■			■		■	■	

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