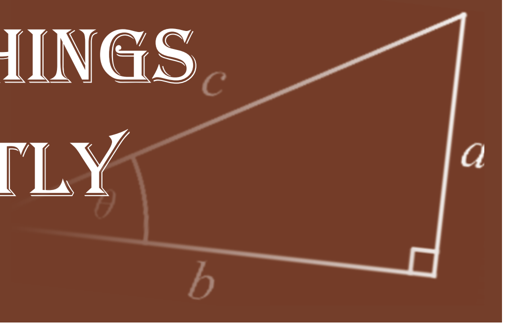
CONNECTING THINGS UP COHERENTLY



By **Colin Foster**

A mathematics teacher told me about a sixth-form (age 16-18) lesson on proof in which they asked the students how they would prove Pythagoras' Theorem. One student said that they would begin with the trigonometric identity $\sin^2\theta + \cos^2\theta \equiv 1$ and apply it to the triangle shown in Figure 1:

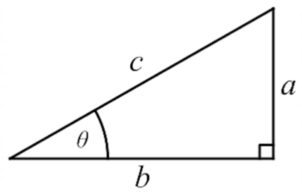


Figure 1. A right-angled triangle.

Their proof was (Note 1):

$$\sin^2\theta + \cos^2\theta \equiv 1,$$
$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1,$$
$$a^2 + b^2 = c^2.$$

The teacher's reaction was that this 'felt backwards', since the trigonometric identity $\sin^2\theta + \cos^2\theta \equiv 1$ that the student was beginning with was, in the teacher's thinking, itself derived from Pythagoras' Theorem, by writing those exact steps in reverse order (Note 2). He asked the student, "Where do you think $\sin^2\theta + \cos^2\theta \equiv 1$ comes from?" The student said that they didn't know – they 'just knew' it, or perhaps it 'came from' the formula book.

This got me thinking about 'what comes from what' in school mathematics, and what it can mean to *prove* things in a system where we haven't explicitly agreed where we are starting from and what counts as 'known' or 'not yet known' at any stage. In university mathematics, we (ideally) build up systematically from definitions and axioms that lead to theorems, which in turn lead to new theorems that build on what's gone before. A move is legitimate if and only if it builds on something we have already established – otherwise, we learn to act as if we are totally ignorant of it. But school mathematics was never intended to be as formal as this. We take a lot of things as 'axiomatic' and perhaps often don't even state them. We might sometimes introduce things which we don't want to prove until later (e.g., see Foster, 2015), and, when we do prove something, we often feel free to use whatever we can find lying around, without necessarily thinking carefully about what those things might themselves be dependent on.

People often talk about the 'connections' in school mathematics, and the importance of helping students to understand the relationships between the different things that they learn. So, seeing that 'there is a connection' between $\sin^2\theta + \cos^2\theta \equiv 1$ and $a^2 + b^2 = c^2$ is surely very important. What the student has done in bridging from the first one to the second one is helpful. But 'connection' perhaps suggests that either direction is equally fine – you can come at it from either end. And does this perhaps work against a sense of building up a coherent mathematical structure of ideas? Might this even lead to unintentional 'circularity'?

There are of course other ways to prove $\sin^2\theta + \cos^2\theta \equiv 1$ that are less obviously directly building on Pythagoras' theorem. For example, a student could begin with

$$\tan^2\theta + 1 \equiv \sec^2\theta$$

(again, 'from the formula book', perhaps), and multiply through by $\cos^2\theta$. As before, this invites the question, "Well, how do we know that $\tan^2\theta + 1 \equiv \sec^2\theta$?", as this is normally derived by beginning with $\sin^2\theta + \cos^2\theta \equiv 1$ and *dividing* by $\cos^2\theta$, so another example of circularity. Alternatively, a student might begin with the equation of a unit circle of radius r, centred on the origin, $x^2 + y^2 = 1$, and substitute in $x = \cos\theta$ and $y = \sin\theta$. But, that circle equation was presumably derived from Pythagoras' Theorem in the first place.

However, there are proofs of $\sin^2\theta + \cos^2\theta \equiv 1$ which track back less obviously to Pythagoras' Theorem. Below are three which seem to be more steps removed (Note 3), although they all depend on things that would certainly be taught much later than $\sin^2\theta + \cos^2\theta \equiv 1$ itself.

1. *Using the compound angle formula $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$ in reverse*

Here, we write:

$$\cos^2\theta + \sin^2\theta \equiv \cos\theta\cos\theta + \sin\theta\sin\theta$$
$$\equiv \cos(\theta - \theta) \equiv \cos 0 \equiv 1.$$

Of course, compound angle formulae would be taught *after* $\sin^2\theta + \cos^2\theta \equiv 1$, so this is 'backwards', in terms of building up the structure. But it could be a nice opportunity for a backward glance to $\sin^2\theta + \cos^2\theta \equiv 1$ from a different starting point.

2. Using differentiation

In a similar way, calculus proofs would certainly come later than students' first encounter with $\sin^2\theta + \cos^2\theta \equiv 1$.

Let $f(\theta) = \sin^2\theta + \cos^2\theta$. Then, using the chain rule,

$$f'(\theta) = 2\sin\theta\cos\theta - 2\cos\theta\sin\theta = 0,$$

meaning that $f(\theta)$ must be a constant (since its derivative is identically equal to zero). So, we can just evaluate $f(\theta)$ at any convenient value of θ, and whatever answer we get must be what $f(\theta)$ is equal to *everywhere*. Since $f(0) = \sin^2 0 + \cos^2 0 = 1$, then $\sin^2\theta + \cos^2\theta$ must *always* equal 1.

3. Using complex numbers

This one is even more advanced:

$$\begin{aligned}\cos^2\theta + \sin^2\theta &= \cos^2\theta - i^2\sin^2\theta \\ &= (\cos\theta)^2 - (i\sin\theta)^2.\end{aligned}$$

Then, by the difference of two squares, this gives

$$\begin{aligned}(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) &= e^{i\theta}e^{-i\theta} \\ &= e^0 \\ &= 1.\end{aligned}$$

I have often felt a sense of this kind of 'circularity' when teaching the calculus of the exponential and logarithmic functions. A teacher might 'prove' that e^x is its own derivative by writing

$$\begin{aligned}y &= e^x, \\ x &= \ln y, \\ \frac{dx}{dy} &= \frac{1}{y}, \\ \frac{dy}{dx} &= y = e^x.\end{aligned}$$

But, how do we know that the derivative of $\ln x$ is $\frac{1}{x}$? Wouldn't that come *after* knowing that $(e^x)' = e^x$? Indeed, might that not be proved as follows.

$$\begin{aligned}y &= \ln x, \\ x &= e^y, \\ \frac{dx}{dy} &= e^y \\ \frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x},\end{aligned}$$

which is just the same kind of set of statements, written in the opposite order. *Either* of these could be a valid proof, but not *both*, surely? We are showing that these results are *consistent* with one another, but are they actually *true*? How can we know that, if we have to assume one to prove the other?

There are didactical choices about the order in which these things might come, and I think it isn't obvious which it might be better to establish first

$$(e^x)' = e^x \text{ and } \int e^x\,dx = e^x + c,$$

or $(\ln x)' = \frac{1}{x}$ and $\int \frac{1}{x}dx = \ln|x| + c$.

Here are some possible trajectories I've seen.

1. Starting with compound interest

The result $(e^x)' = e^x$ looks much simpler to write down than $(\ln x)' = \frac{1}{x}$, and involves powers, rather than logarithms, and so may seem like it should be the more accessible starting point. So, students might begin in a 'compound interest' kind of scenario, with money, say, growing by smaller and smaller amounts accrued more and more frequently:

Invest £1 at a rate of interest of 100% per year, and after a year you have

£$(1 + 1) =$ £2.

Invest £1 at a rate of interest of $\frac{1}{2}$ of the total, twice a year, and after a year you have

£$\left(1+\frac{1}{2}\right)^2 =$ £2.25.

Invest £1 at a rate of interest of $\frac{1}{3}$ of the total, three times a year, and after a year you have

£$\left(1+\frac{1}{3}\right)^3 =$ £2.37.

Eventually, if you get your interest paid daily, you will obtain

£$\left(1+\frac{1}{365}\right)^{365} =$ £2.71.

Students see the sequence

$1, \left(1\frac{1}{2}\right)^2, \left(1\frac{1}{3}\right)^3, \left(1\frac{1}{4}\right)^4, \left(1\frac{1}{5}\right)^5, \ldots$

appearing to converge to a value near to 2.718… Then, we can assume that this limit exists, and *define* it for *continuously compounded interest* as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n,$$

together with the related *function*

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n,$$

where x is the 'annual interest rate'. This seems a valuable and important activity, but it isn't particularly easy to go on to use this definition to prove that $(e^x)' = e^x$. So, while I like this, it probably isn't my preferred approach for introducing e^x as its own derivative.

2. Starting with the graphs of $y = a^x$

Another way in is to have students do some plotting of $y = a^x$ for different $a > 0$, and estimate and draw gradient functions, either on paper or using graph-drawing software. This activity strongly suggests that there exists a graph somewhere between $y = 2^x$ and $y = 3^x$ with the very special property that the gradient at every point is equal to the y-value. (In dynamic geometry software, slide along the x-axis a right-angled triangle with unit base, and its top vertex on the curve, and its hypotenuse will lie on the tangent at every point.) The function we seek has a rate of increase equal to however much of it you've got at that point – a good way to think about what exponential growth entails (Note 4). Students can narrow down the value to 2.7ish, and then we can just say, "The value is actually 2.71828…, and we call it e". There is no proof of convergence here, but then neither was there in the $\left(1+\frac{1}{n}\right)^n$ approach.

This is perhaps a bit like taking $(e^x)' = e^x$ as our *definition* of e^x, and so then we have nothing to prove – we merely have to demonstrate plausibility. We say that e^x is the unique function (up to a multiplicative constant) that is its own derivative. Students should have developed some intuition through this task that there might be a number a such that $(a^x)'=a^x$, and it is plausible that it should have a value close to the value of e, but nothing rigorous has been done. We can then use this result to establish that

$\int e^x\,dx = e^x + c$, $(\ln x)' = \frac{1}{x}$, $\int \frac{1}{x}dx = \ln|x| + c$.

3. Starting with the Taylor series

The easiest way to show that e^x is its own derivative is to take as its definition the power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

together with the idea that differentiating term by term is allowed. This builds nicely on what students already know – how to differentiate *polynomials* containing terms like x^n – although, of course, a power series is an *infinite series*, and not a polynomial. At the same time, or later, you can use the power series for $\sin x$ and $\cos x$ (Note 5) to show the relationships portrayed in Figure 2, which perhaps nicely avoids other methods of proving these that involve fiddly trigonometric identities and limits.

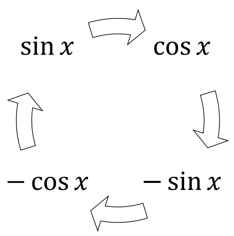


Figure 2. Each arrow represents differentiation with respect to x

Personally, I like this approach, and building up these series term by term in graph-drawing software can be quite magical. Teachers sometimes object that this is 'circular', because the Taylor series themselves are derived by assuming that $(e^x)' = e^x$ and $(\sin x)' = \cos x$? But I think this is only a problem if you don't really buy into treating the Taylor series as the *definition*.

With any of these approaches, once you have that $(e^x)' = e^x$, you can get to

$$\int \frac{1}{x}dx = \ln|x| + c$$

by saying that if

$$\frac{dy}{dx} = \frac{1}{x}$$

then

$$\frac{dx}{dy} = x,$$

and we know that the function $x(y)$, whose derivative is itself, is $x = Ae^y$. So, rearranging, $y = \ln\left(\frac{x}{A}\right)$, or $y = \ln x - \ln A = \ln x + c$, where c is a constant.

There is then a bit of fussing about the domain, because $\ln x$ is defined only for $x > 0$. Sometimes this seems to be addressed by saying "Well, we'd better put modulus signs in, to make sure that taking the logarithm doesn't give us an error!" But, the fact that putting in modulus signs means that we get *an* answer doesn't, of course, mean that it gives us the *right* answer! So, we really need to consider the two cases separately.

- If $x > 0$, then, in $x = Ae^y$, $A > 0$. So, $\ln\left(\frac{x}{A}\right)$ can be written as $y = \ln x - \ln A = \ln x + c$, where c is a constant.
- However, if $x < 0$, then, in $x = Ae^y$, $A < 0$. So, although $y = \ln\left(\frac{x}{A}\right)$ is still fine, because $\frac{x}{A} > 0$ (the quotient of two negative numbers is positive), when we split this up into a *difference* of logarithms, we need to treat it as $\ln\left(\frac{-x}{-A}\right)$, and write $\ln(-x) - \ln(-A)$, which can be written as $y = \ln(-x) + c$, where c is a (different) constant.

So, all of this can be summarised by writing $y = \ln|x| + c$.

4. First teach $\int \frac{1}{x}dx$

Finally, there is the possibility of *beginning* with $\int \frac{1}{x}dx = \ln|x| + c$. In more advanced work, we *define* the logarithm function as

$$\ln x = \int_1^x \frac{1}{t}dt,$$

so perhaps it might not be such a bad idea to begin at this end, rather than with e^x, and work in the opposite direction?

Early on, when teaching

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1,$$

inevitably, something has to be said about the division by zero that would happen if $n = -1$. It is possible to leave this as a loose thread to pick up on later, and say: "We'll come back to this later and find out what happens for this exceptional case". I've usually treated it that way. Alternatively, it could be dealt with now and lead on to the calculus of the logarithmic and exponential functions.

Once the chain rule for differentiation has been learned, we can note that, since constants can be taken outside the integral,

$$\int \frac{1}{ax}\,dx = \frac{1}{a}\int \frac{1}{x}\,dx.$$

This means that we can write

$$\int_1^x \frac{1}{at}\,dt = \frac{1}{a}\int_1^x \frac{1}{t}\,dt.$$

So, if we suppose that the integral that we desire is a function f, such that

$$\int_1^x \frac{1}{t}\,dt = f(x), \text{with } f(1) = 0,$$

then, by the chain rule,

$$\frac{1}{a}\big(f(ax) - f(a)\big) = \frac{1}{a}\big(f(x) - f(1)\big).$$

Since $f(1)=0$ (the upper and lower limits on the integral coincide), simplifying this gives

$f(ax)=f(x)+f(a)$, for all values of $x\geq 1$.

Our mystery function f has the property that *multiplying* x by a constant a has the effect of *adding* a constant ($f(a)$) onto $f(x)$. So, we need a graph for which a *scaling* in the x-direction is equivalent to a vertical *translation*. Not many graphs do that. The 'multiplication leading to addition' feature is highly suggestive of the logarithm function. This is no more than a nudge, and certainly we have no reason to think that it will turn out to be the *natural* logarithm, but it's perhaps a possible way to address this.

But I think overall my preference is to start with the power series (see also Ullah, Aman, Wolkenhauer, & Iqbal, 2021). I would be very interested to know what any readers think.

Notes

1. The transition from $\sin^2\theta + \cos^2\theta \equiv 1$, with the identity symbol to $a^2 + b^2 = c^2$, with an equals sign, is interesting here (see Foster, 2021). If a, b and c are constrained to be the sides of a right-angled triangle, with c the hypotenuse, then $a^2 + b^2 = c^2$ is true for *all* possible values of a, b and c, so shouldn't it be $a^2 + b^2 \equiv c^2$? If you disagree, then which symbol, = or ≡, do you think should be used on line 2 of the proof, for

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1\ ?$$

2. An issue with this, of course, is that $\sin^2\theta + \cos^2\theta \equiv 1$ holds for *all* real θ, whereas the right-angled triangle proof deals only with the case where $0 < \theta < \frac{\pi}{2}$.

3. For these and other proofs, see **https://math.stackexchange.com/questions/607103/prove-sin2-theta-cos2-theta-1**.

4. It is interesting how often the word 'exponentially' is misused, for example by politicians, to mean 'a lot'. But an exponential growth can be 'small' and still exponential (see Suri, 2019). Radioactive decay is exponential, but, for isotopes with half lives in the millions of years, it is unimaginably slow on a human timescale.

5. Rather than *Taylor series*, we should perhaps call these the *Madhava series*: **https://en.wikipedia.org/wiki/Madhava_series**.

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