CROSSING OUT

by **Colin Foster**

In mathematics, crossing something out doesn't necessarily mean that you've made a mistake. I'm not just thinking of the 'not' symbols, like ≠ and ∉. I have in mind processes where it's normal to indicate a change in such a way that what was previously present is still clearly visible.

This is very common with exchanging in the subtraction algorithm; e.g.:

$$\begin{array}{rccc} & & {\scriptstyle 8} & {\scriptstyle 1} \\ & 1 & \cancel{9} & 5 \\ - & & 2 & 7 \\ \hline & 1 & 6 & 8 \\ \hline \end{array}$$

or with 'cancelling down' when multiplying fractions [Note 1]:

$$\frac{\cancelto{1}{2}}{\cancelto{3}{15}} \times \frac{\cancelto{1}{5}}{\cancelto{3}{6}} = \frac{1}{9}$$

However, I wonder if there are more occasions where crossing things out would be valuable. This could save time, because less writing is needed, but it could also help to avoid the *split-attention effect* (Sweller, Ayres, & Kalyuga, 2011; see also Chen, Castro-Alonso, Paas, & Sweller, 2018), where eyes have to flick back and forth between two different places, searching for the connections, which we know generally taxes working memory more than including all the information in one place.

For example, although crossing out is widely accepted when *multiplying* fractions and 'cancelling down', I find that it tends to be frowned on when *adding or subtracting* fractions to indicate what I sometimes like to call 'cancelling up':

$$\frac{\cancelto{6}{2}}{\cancelto{15}{5}} + \frac{\cancelto{5}{1}}{\cancelto{15}{3}} = \frac{11}{15}$$

To me, this notation clarifies rather nicely that we are replacing $\frac{2}{5}$ by $\frac{6}{15}$ and $\frac{1}{3}$ by $\frac{5}{15}$, and it is consequently very easy to check that I haven't gone wrong. For the same reason, it's also easier to find errors when marking if students lay it out like this, precisely because it's a clearer communication of what's happening. However, I find that teachers tend to object to doing this, and I'm not exactly sure why. They sometimes say that it's 'messy', but how is it any messier with addition than it is with multiplication? In each case, you are crossing out and replacing either two or four numbers (for a two-fraction calculation), so why is this more of a problem for addition than for multiplication? They say that it is 'too condensed' – but too condensed for what, exactly?

I find that people sometimes object that although this layout might be quicker and easier, it is also more error prone. But I don't see why that should be the case, and I think I would actually argue the opposite. If, instead of using crossing out, I do the more usual process of writing:

$$\frac{2}{5} + \frac{1}{3} = \frac{6+5}{15} = \frac{11}{15},$$

it is much harder to check at a glance that the middle, red part is correct. The quick way to do this is to invoke 'cross-multiplying', which I am not a huge fan of, using mental images like

which seem to detract from the simplicity of replacing the original pair of fractions with a pair of fractions equivalent to each.

A fuller version of this layout, where we write:

$$\frac{2}{5} + \frac{1}{3} = \frac{6}{15} + \frac{5}{15} = \frac{11}{15}$$

is clearer about this, but checking still involves looking across from $\frac{2}{5}$ to $\frac{6}{15}$, jumping over the '$+\frac{1}{3}=$' symbols in between, and, of course, this is even worse with more than two fractions to add. (Writing the common denominator of 15 *twice* perhaps clarifies that it is the denominator of *both* fractions, but also perhaps makes it slightly more likely that the student will incorrectly add them to get 30.)

Lining up vertically helps:

$$\begin{aligned} &\frac{2}{5} + \frac{1}{3} \\ =\ &\frac{6}{15} + \frac{5}{15} \\ =\ &\frac{11}{15} \end{aligned}$$

but you still have to jump down from the numerator in line 1 to the numerator in line 2, passing over the denominators, and so on, whereas with the crossing-out layout the numbers are as close to those they are replacing as they could possibly be. This seems optimal to me.

Maybe these seem like small considerations. With our 'curse of knowledge', we may be tempted to think that any of these layouts should be fine, and why should we overthink it? But, for students whose working memory is getting clogged up during this process, adopting even a slightly more transparent layout could be the difference between success and failure. And, indeed, for anyone who might potentially make an error (and that includes all of us), it is surely better to use layouts that make this less likely rather than more likely. There has been increasing attention recently given to *Cognitive Load Theory* (Sweller, Ayres, & Kalyuga, 2011) in mathematics teaching, but this has tended to focus on improving teacher explanations and presentations. An equally important task might be to evaluate competing *methods* or *styles of layout* according to split-attention or other relevant effects.

For those who are with me this far, maybe it's worth pushing into what I suspect are some more controversial examples. Should we, for instance, tolerate things like:

$3x+\cancel{5}=\overset{9}{\cancel{1}4}$?

or perhaps even

$3(x-\cancel{2})+\cancel{6}=9$?

Clearly, there does come a point where the crossings out become unwieldy and potentially confusing; in particular, one thing you can't easily do with a crossing-out layout is cross anything out when you make a mistake! However, I don't think that means we should dismiss these approaches entirely. I'm not sure that crossings out such as those above are really any different from the practice of, say, dismissing a null sequence by writing:

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)=\lim_{n\to\infty}\left(\frac{n+1-1}{n+1}\right)=\lim_{n\to\infty}\left(1-\cancelto{0}{\frac{1}{n+1}}\right)=1.$$

Set against concerns about messiness and the risk of errors, it is worth noting that many student errors arise when copying unmodified parts of one line onto the line below, where factors or terms get lost or garbled. For example, on line 1 you might have something like:

$$3\left(\frac{5x}{x+1}-\frac{2}{x}\right)-4x$$

but on line 2 either the factor of 3 or the $4x$ term has dropped out, or $-4x$ has become $\times 4x$, because all the student's attention has been on sorting out the details within the bracket, and everything else has got forgotten. If every time a new line is written there is, say, a 10% chance of a copying error of some kind, then minimising the number of copied lines needed would seem sensible (since $0.9^n \approx 0$ for even medium-sized n).

Finally, I wonder if one of the reasons that crossing out is unpopular is that it's inconvenient for textbook designers and typesetters to render on the page. If so, that seems to me not a good thing to have dictating our practice and preferences. Crossing things out doesn't have to be messy, and sometimes I think it can be the clearest and least error-prone way to illustrate and keep track of what's going on. But, if you disagree, please write in and make your case!

Note

1. There are also less-common instances, such as when performing Dijkstra's algorithm by hand and updating a node with a new, shorter distance.

References

Chen, O., Castro-Alonso, J. C., Paas, F., & Sweller, J. 2018 Extending cognitive load theory to incorporate working memory resource depletion: evidence from the spacing effect. *Educational Psychology Review*, *30*(2), 483-501.

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