Differentiating Inverse Trigonometric Functions

by **Colin Foster**

There are many tempting patterns in mathematics that don't work out the way one might expect (Foster, 2020). For example,

> *When you differentiate sin x you get cos x.*
> *So, when you differentiate* <u>*inverse*</u> *sin of x, you will get …?*

Well, the answer is not quite $\cos^{-1} x$; in fact, you get $\frac{1}{\sqrt{1-x^2}}$.

How strange is that? Confronted with this, a sixth-form pupil asked, "Where did the sine go?" It seems reasonable that in calculus a sine might transform into something like a cosine, but how can it just disappear altogether, without leaving a trace? It would be like differentiating $\sin x$ and getting x^3.

Thinking about the pupil's prior experiences, there is perhaps quite a sharp line in school mathematics between solving 'trigonometric equations', such as $10\sin x - 3\cos^2 x = 5$, and solving *non*-trigonometric equations, such as $3x^2 + 10x = 8$. (This is often reinforced by using θ rather than x for the unknown in the former.) There is no way that manipulation of a trigonometric equation will ever lead to all the trigonometric functions dropping out to leave 'just xs'. That would be a sure sign that something had gone haywire with the algebra, such as cancelling the 'sin' in an expression like $\frac{\sin x}{\sin y}$ to give $\frac{x}{y}$.

One way to respond could be by addressing the pupil's false-but-tempting idea that, in general,

$$f'\left(f^{-1}(x)\right) = f^{-1}\left(f'(x)\right).$$

Although this may sound plausible when expressed verbally, the symbols make it look unlikely, and it is easy to find simple counterexamples, such as $f(x) = x^2$ for $x \geq 0$, where $f'\left(f^{-1}(x)\right) = f'\left(\sqrt{x}\right) = \frac{1}{2}x^{-\frac{1}{2}}$ and

$$f^{-1}\left(f'(x)\right) = f^{-1}(2x) = \frac{1}{2}x.$$

However, $\sin x$ is itself a perfectly good counterexample, so perhaps we should return to this.

Proving that something is the case doesn't necessarily offer any insight as to why. The standard moves here are:

$$y = \sin^{-1} x \Leftrightarrow x = \sin y$$

$$\frac{dx}{dy} = \cos y$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

Each step may be OK [Note 1], but the overall result still seems mysterious. As the pupil said in the ensuing conversation, "How can the answer not have anything to do with sine?" There is such a beautiful relationship where the gradient function of sine is cosine, which students can see as intuitively plausible by examining the sine graph and thinking about where the gradient is positive, negative or zero, and where it is increasing and decreasing. It seems very weird that, once you reflect (part of) this graph in $y = x$ (which is all that $\sin^{-1}$ is doing), the gradient function now has nothing at all to do with sine or cosine [Note 2].

This question also seems reasonable when you think about the prior work on differentiation that had led up to this lesson. When you differentiate $e^{\text{something}}$, your answer is going to contain $e^{\text{something}}$. Differentiating polynomials gives polynomials; differentiating trigonometric functions gives trigonometric functions. Each class of function fits nicely into a different chapter of the textbook, so what is going on with differentiating $\sin^{-1} x$? Why is the sine dropping out?

An analogous thing happens with *integration* when integrating x^n, where the result is $\frac{x^{n+1}}{n+1} + c$, for all n except $n = -1$. The fact that $\int x^{-1} dx = \ln|x| + c$ is quite a shock, and takes a bit of work to make sense of. The surprise is that we are working happily within the powers of x, when suddenly we are 'hit by a log', so to speak! Perhaps there is a similar potential query about why differentiation of $\ln x$ produces $\frac{1}{x}$, rather than anything involving logarithms?

Similar surprises arise in integration when pupils are confronted with the difference between, say,

$$\int \frac{1}{x^2 - 1} dx = \int \left(\frac{1/2}{x-1} - \frac{1/2}{x+1} \right) dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c$$

and $\int \frac{1}{x^2+1}dx = \tan^{-1}x + c.$

One moment, we are in the world of partial fractions and logarithms; the next, with just a single sign change, we have switched to trigonometric functions! Again, proving this result, by suddenly deciding (for no apparent reason) to do a substitution in which $x = \tan u$, certainly shows us that the result is correct, but doesn't really help us to see where it came from:

$$\begin{aligned}\int \frac{1}{x^2+1}dx &= \int \frac{1}{\tan^2 u+1}\sec^2 u\,du\\ &= \int du\\ &= u+c\\ &= \tan^{-1}x + c.\end{aligned}$$

This feels like a lucky fluke, caused by the tangent function happening to differentiate to make $\sec^2$, while, at the same time, purely by accident, $\sec^2$ happens (for unrelated reasons, to do with $\sin^2 + \cos^2 \equiv 1$) to be equal to $\tan^2 + 1$, so it all cancels out nicely. Is there more to understand here than this?

This time, perhaps there is, because, if we know about imaginary numbers, we *can* express $\frac{1}{x^2+1}$ in partial fractions, just like we did with $\frac{1}{x^2-1}$:

$$\begin{aligned}\int \frac{1}{x^2+1}dx &= \int \left(\frac{\frac{1}{2i}}{x-i} - \frac{\frac{1}{2i}}{x+i} \right) dx\\ &= \frac{1}{2i}\ln|x-i| - \frac{1}{2i}\ln|x+i| + c\\ &= \frac{1}{2i}\ln\left|\frac{x-i}{x+i}\right| + c\\ &= \frac{i}{2}\ln\left|\frac{x+i}{x-i}\right| + c,\end{aligned}$$

which means that $\tan^{-1}x = \frac{i}{2}\ln\left|\frac{x+i}{x-i}\right|.$

So, perhaps polynomials, logarithms and trigonometric functions are more closely related than we tend to think; consider, for example,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ ,$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}\ ,$$
$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The boundaries between these categories of functions are more blurred than we might have thought.

Returning to $\sin^{-1}x$, I am indebted to Bob Burn for alerting me to the possibility of the diagram shown in Fig. 1, in which the vertical side of the large right-angled triangle is x, meaning that the opposite angle in that triangle must be $\sin^{-1}x$, and therefore the arc length that subtends this angle must also be $\sin^{-1}x$. Now, creating a tangent to the circle where it meets the hypotenuse of this triangle, and looking for dy and dx, we can see, by similar triangles, that, in the limit, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

So perhaps we were wrong to think that $\frac{1}{\sqrt{1-x^2}}$ did not involve (co)sine, since it involves x, which is a sine of something. Does this go some way towards answering

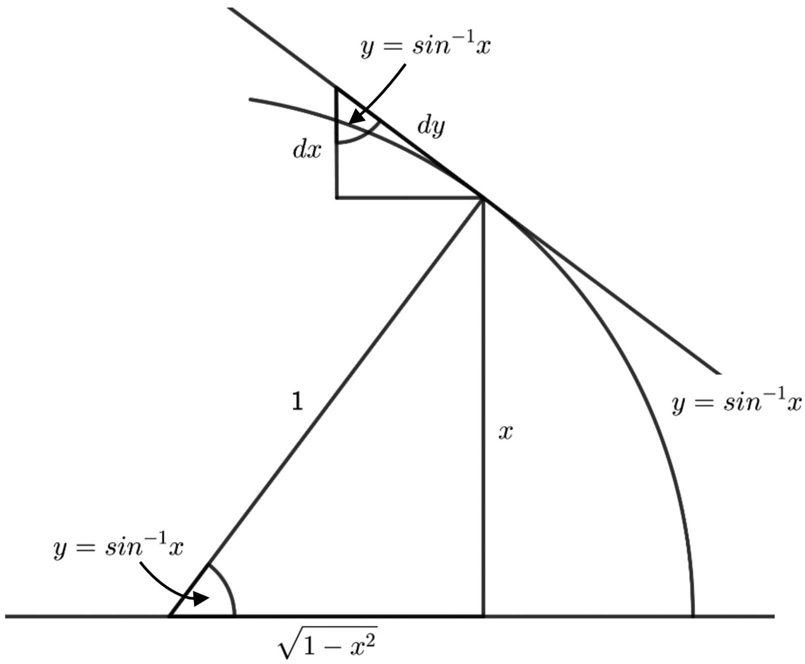

Fig. 1 Differentiating $y = \sin^{-1}x$ geometrically

the pupil's question? I would be very interested if any reader can do better.

Note

1. Some thought needs to be given to why we take the *positive* square root. Looking at the graph $y = \sin^{-1} x$ (Fig. 2), we can see that the gradient of this function is positive everywhere (in fact, it is never less than 1), so taking the positive root ensures we have the correct gradient function. Another way to see this is to note that $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$, and $\cos y$ is never less than zero throughout the range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

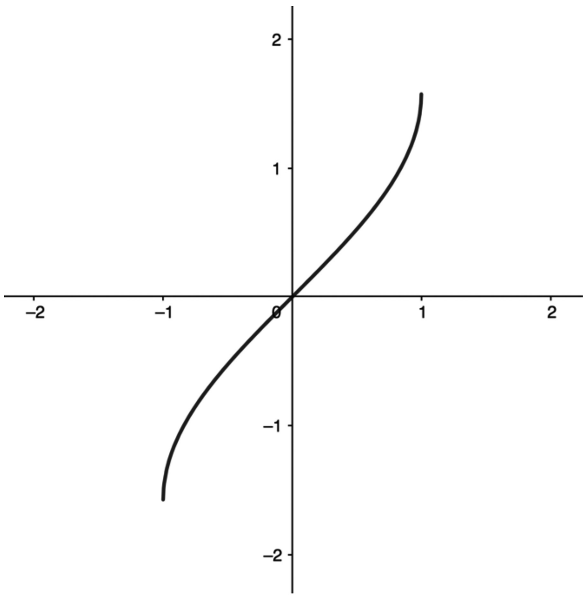

Fig. 2 $y = \sin^{-1} x$

2. If we stubbornly insisted on expressing $\frac{dy}{dx}$ in terms of $\cos^{-1} x$, then we could do this, in a highly-contrived fashion, by writing

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos\left(\sin^{-1}\left(\cos\left(\cos^{-1} x\right)\right)\right)}$$ but this doesn't seem to offer me any insight.

Acknowledgement

I would like to thank Dr Bob Burn for very stimulating conversations on this question.

Reference

Foster, C. (2020). Trusting in patterns. *Mathematics in School* **49**, 3, pp.17–19.

Keywords: Calculus; Differentiation; Integration; Inverse functions; Trigonometric functions.

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Correction and apology
(Colin Foster's article in the Nov. 2020 issue)

Due to some infelicitous handling by the Editors of a small section of Colin Foster's article 'Differentiating inverse trigonometric functions' (*Mathematics in School* **49**, 5, pages 10–12), the message that the author wished to convey was lost. The simplest way to correct the situation is to replace the start of the fourth paragraph of the first column of page 10, with:

> One way to respond could be by addressing the pupil's false-but-tempting idea that in general,
>
> $$(f^{-1}(x))' = (f'(x))^{-1}$$

and the subsequent algebra for $f(x) = x^2$ with:

$$(f^{-1}(x))' = (\sqrt{x})' = \frac{1}{2}x^{-\frac{1}{2}}$$

$$(f(x)')^{-1} = (2x)^{-1} = \frac{1}{2}x.$$

Here, of course, Colin is using the notation f^{-1} to indicate 'the inverse function of f', not f raised to the power of –1,

so $(2x)^{-1} = \frac{1}{2}x$, rather than $\frac{1}{2x}$.

The Editors apologise for this lapse and hope it did not spoil your enjoyment of the article.