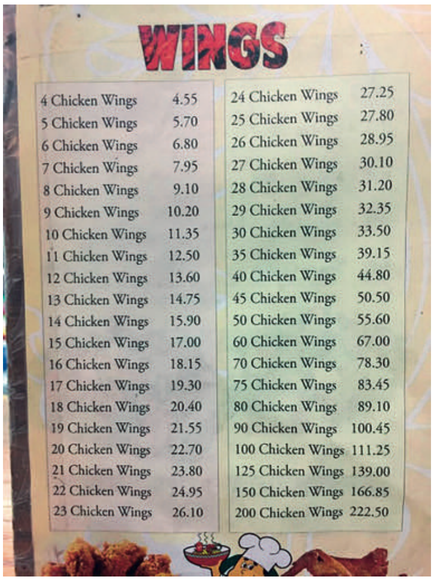
METHODS THAT ARE JUST MENTAL CLUTTER

By **Colin Foster**

The physicist Frank Wilczek (2010) recollects being given a textbook containing a chapter entitled “Ohm’s Three Laws”. He was familiar with Ohm’s law, that $V = IR$, where V is the voltage, I is the current and R is the resistance, but “was very curious to find out what Ohm’s other two laws were ... I soon discovered that Ohm’s second law is $I = V/R$. I conjectured that Ohm’s third law might be $R = V/I$, which turned out to be correct” (Wilczek, 2010, p. 24).

This appears to be a good example of giving a person a fish (or three fishes, in this case) rather than a *fishing rod*, which would enable them to catch all the fish they need for themselves. Teaching $V = IR$, plus the ability to rearrange equations, would seem to be more economical than teaching $V = IR$, $I = V/R$ and $R = V/I$ as three separate ‘laws’, and I sometimes use the phrase “Ohm’s three laws” as a metaphor for teaching unnecessary mental clutter like this. The ability to rearrange equations gives “a lot for a little” (see Hewitt, 2017), and is something students have to learn anyway, so why teach two additional facts which can be easily derived from the first one? This would be like teaching Pythagoras’ Theorem to find the length of a hypotenuse ($a^2 + b^2 = c^2$) and Pythagoras’ Theorem for a leg ($c^2 - b^2 = a^2$) as separate entities.


WINGS

4 Chicken Wings	4.55	24 Chicken Wings	27.25
5 Chicken Wings	5.70	25 Chicken Wings	27.80
6 Chicken Wings	6.80	26 Chicken Wings	28.95
7 Chicken Wings	7.95	27 Chicken Wings	30.10
8 Chicken Wings	9.10	28 Chicken Wings	31.20
9 Chicken Wings	10.20	29 Chicken Wings	32.35
10 Chicken Wings	11.35	30 Chicken Wings	33.50
11 Chicken Wings	12.50	35 Chicken Wings	39.15
12 Chicken Wings	13.60	40 Chicken Wings	44.80
13 Chicken Wings	14.75	45 Chicken Wings	50.50
14 Chicken Wings	15.90	50 Chicken Wings	55.60
15 Chicken Wings	17.00	60 Chicken Wings	67.00
16 Chicken Wings	18.15	70 Chicken Wings	78.30
17 Chicken Wings	19.30	75 Chicken Wings	83.45
18 Chicken Wings	20.40	80 Chicken Wings	89.10
19 Chicken Wings	21.55	90 Chicken Wings	100.45
20 Chicken Wings	22.70	100 Chicken Wings	111.25
21 Chicken Wings	23.80	125 Chicken Wings	139.00
22 Chicken Wings	24.95	150 Chicken Wings	166.85
23 Chicken Wings	26.10	200 Chicken Wings	222.50

Figure 1 Chicken wings price list circulated on Twitter
(see www.insider.com/restaurants-pricing-confusing-math-2018-10).

We often come across situations in which a short, simple rule could replace a multitude of redundant information. There is a cafe near me where the staff use a laminated ready reckoner next to the cash register that shows them the total cost of n items priced at £2 each, and this reminds me of the rather mysterious chicken wings price list that circulated recently on *Twitter* (see Figure 1, Note 1). It feels like an extraordinarily inefficient way of obtaining a price.

But, are objections to things like this just ‘the curse of knowledge’ talking? I am lucky enough after years of practice to be able to rearrange equations very easily in my head, so why would I bother remembering rearranged versions of such a simple equation as Ohm’s (first) law? But, students are not necessarily in this position, and they often seem to really like mnemonics like formula triangles, even though their teachers frown on them (see Foster, 2021). Whether something is worth remembering depends on how easily you can derive it and how often you need to access it. It is not enough to just say, “You can work it out from such-and-such and so you don’t need to remember it”, if actually that working out process is lengthy or difficult in practice, and you are never sure whether you have done it right or not. In the case of a more complicated equation, such as the cosine rule, $c^2 = a^2 + b^2 - 2ab\cos C$, maybe it would not be ridiculous to also remember the version for calculating angle C:

$$C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

as there are a few more steps to get from one of these to the other (see Note 2). Personally, I still find that this gap is manageable in my head, so I don’t feel the need to remember both versions. But this time it doesn’t seem quite so ridiculous to do so, whereas it would seem strange to me to, say, remember separately the different versions with the letters permuted:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$

which would be horribly easy to muddle up, given that they can be so easily obtained by cyclically permuting a/A, b/B, c/C. (Alternatively, you can just always use one formula and relabel the triangle when necessary.)

Reflecting a bit more, I find that when I am applying the cosine rule to a specific triangle, I don't actually think in terms of letters at all. I actually write down the relation using some mental process like "the side that's opposite the angle, squared, equals the sum of the squares of the other two sides, minus twice the product of those two sides multiplied by the cosine of the angle between them". I haven't memorised the cosine rule using those specific words, but that tends to be how I think about it when using it. And so I am not troubled if the sides happen to be labelled with something awkward like *b, c* and *d*. I wouldn't need to think 'Let my a = their b', and so on. Similarly, I think I use the formula for the area of a triangle, Area = $\frac{1}{2}ab\sin C$ as "half the product of the sides, multiplied by the sine of the angle between them", rather than as letters, but I also know it as letters, as, if asked to write it down, I would certainly use those traditional *a, b* and C letters (see Note 3). So it isn't as simple as having one preferred form that I remember and use.

The more I think about it, the less straightforward it is to decide what counts as unnecessary mental clutter, as everything except definitions can be derived from something simpler. Some people feel that in calculus the quotient rule for differentiating $\frac{u}{v}$ comes into this category, and instead of using it they just use the product rule on uv^{-1}, whereas other people would say that this is 'like deriving the quotient rule every time'. Many years ago, I used to remember all of the product-sum trigonometric identities, like

$$\sin P + \sin Q \equiv 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

and I still roughly know them, but I use them so rarely that I no longer know them quite confidently enough to use. Now, I would work them out from $\sin(A+B) + \sin(A-B)$, and, on a good day, I suppose I could just about do all of that in my head – but I would actually probably write it down just to be sure. There is still some remembering here, of course, because I need to remember that the 'trick' is to add $\sin(A+B)$ and $\sin(A-B)$ and expand those out, and of course I need to know those expansions, but, because I do, I no longer feel that I need to have the product-sum ones immediately available.

Some rules are even worse than redundant baggage, because they turn out to be 'rules that expire' (Dougherty, Bush, & Karp, 2017), meaning that they only work in a limited (and perhaps sometimes unspecified) range of situations. I don't think we should criticise a method because it doesn't 'always' work, because there's a sense in which very little 'always' works. All methods work in the situations they're designed for, and then break down outside those situations. But, I think it's fair to complain about methods that seem to have extremely limited ranges of usefulness, particularly where the domains of applicability are not clearly specified, and are often highly targeted towards the particular contents of examination specifications.

A more subtle example, that Wilczek's anecdote reminded me of, is the 'reciprocal formulae' in school science, such as the one for the overall effective resistance R of two resistors in parallel, with resistances R_1 and R_2 (see Note 4):

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \qquad (1)$$

Students often struggle with the steps needed to make R the subject of this formula:

$$\frac{1}{R} = \frac{R_1 + R_2}{R_1 R_2}$$

$$R = \frac{R_1 R_2}{R_1 + R_2} \qquad (1')$$

Consequently, they might be told to use the formula (1′) directly, and to remember this as 'the product divided by the sum'.

This again feels like unnecessary mental clutter, because the distance between (1) and (1′) feels so small – there is no need to memorise (1′) if I already know (1). I can immediately see that combining the fractions on the right side of (1) will give $\frac{\text{sum}}{\text{product}}$, and so it feels like 'just one step' from here to invert this to get $\frac{\text{product}}{\text{sum}}$, and so it feels wasteful to remember this. If I ever I was unsure whether it was

$$\frac{R_1 + R_2}{R_1 R_2} \, or \, \frac{R_1 R_2}{R_1 + R_2},$$

I could just repeat this reasoning. Alternatively, I could use dimensions to see that the total resistance must be the latter, since $\frac{\text{resistance}\times\text{resistance}}{\text{resistance}}$ has the dimensions of resistance, whereas $\frac{\text{resistance}}{\text{resistance}\times\text{resistance}}$ doesn't.

Like with Ohm's law, you might say that this argument only applies if you're already good at rearranging equations in your head. But the danger for the student who doesn't yet have this kind of fluency with equations is that, to compensate, they get loaded down with more and more things to remember. This is likely to make them more error-prone, and it also makes it less likely that they will develop the kind of facility with equations that they need, if they are constantly avoiding the repeated practice. They find that mathematics becomes a subject where the further you go the *more* things there are to try to remember, rather than a subject where the further you go the *fewer* things there are to remember. Once you get to the point where you learn De Moivre's Theorem, you can derive lots of trigonometric identities very easily, without having to rely on having memorised others.

However, in this case a more serious objection to $\frac{\text{product}}{\text{sum}}$ is how this might mislead students in a scenario where they have *three* or more resistors in parallel:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

If they have no idea what they are doing, then they are likely to interpret $\frac{\text{product}}{\text{sum}}$ as

$$R = \frac{R_1 R_2 R_3}{R_1 + R_2 + R_3} \qquad (2)$$

rather than the correct formula, which is

$$R = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_1 + R_2 + R_3} \qquad (2')$$

The (2) version looks like it follows the pattern, and seems like the obvious generalisation for an extra resistor, but on closer inspection it has the dimensions of resistance *squared*, rather than resistance. To work for more than two resistors, the rule $\frac{\text{product}}{\text{sum}}$ would need to be expanded to $\frac{\text{sum of all possible pairs of products}}{\text{sum}}$, which is an unnecessarily complicated version to use if you are only even going to meet the 2-resistor case. So, the teacher might be inclined to think, "Well, at this level they never have to deal with more than two resistors, so this is not a problem." But, for me, $\frac{\text{product}}{\text{sum}}$ seems like one of those things that there doesn't seem to be a good rationale for teaching in mathematics. (The case from the point of view of science teaching may be different, of course, since it isn't the science teacher's job to teach rearranging equations.)

So I think the judgment as to whether a method is helpful, or just unnecessary mental clutter, is not always an easy one to make. But we need to go beyond "Will this method be easier for the student to use?", because thinking that way tends to lead us to a proliferation of little methods that are hard to remember and easy to muddle up.

Notes

1. What would happen if you asked for 32 chicken wings?
2. However, I am less comfortable when these are presented as separate 'rules' ('The cosine rule for sides' and 'The cosine rule for angles'), and it seems to be only a curiosity that in fact one can be obtained from the other.
3. Words often *sound* much more complicated than formulae, but can be easier to remember and apply in practice. I think of integration by parts as "the first, times the integral of the second, minus the integral of the derivative of the first, times that same integral". Writing it down it looks impossibly complicated at first, but I personally find it much easier to use than having to write down "Let u = ... and $\frac{dv}{dx}$ = ...", and if you have an example involving awkward letters, like $\int ve^{2v}dv$, it is no problem.
4. The so-called *optic equation* for resistors in parallel is isomorphic to several other important science equations, such as those for inductors in parallel, capacitors in series, and the *thin-lens formula* in optics.

References

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