Minimalist Approaches to Teaching Trigonometry

By **Colin Foster**

I was listening to some mathematics teachers talking about the teaching of trigonometry, and one of them said, “I only teach sine – I don’t bother with cos and tan”. This led to immediate disagreement about what it means to ‘teach trigonometry’: Is our task to teach ‘sin, cos and tan’, or is our task to enable students to be able to solve right-angled triangles efficiently, given any possible combination of information?

The ‘usual’ approach

There are essentially six different configurations when finding the side length of a right-angled triangle, given an angle and one other side (Figure 1).

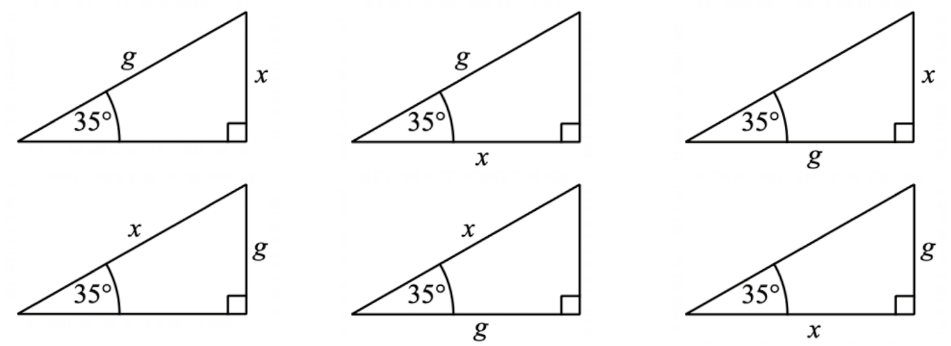


Figure 1. The six configurations of one-side-given (g) and one side required (x) in a right-angled triangle.

In the conventional ‘SOHCAHTOA’ approach (Note 1), students first need to learn to label the sides of the triangle as ‘hypotenuse’, ‘opposite’ and ‘adjacent’. Then they need to determine which of these three quantities is *irrelevant* to the question (i.e., it is neither the quantity *given* nor the quantity *required*), and find the appropriate ratio (sine, cosine or tangent) that connects the *other two*. This can be quite a sophisticated bit of thinking that students do not necessarily find easy. Then they need to set up an equation using this ratio and the two relevant sides, and solve it to find the required side length (Note 2). Each configuration is solved by using one of the three ratios, and either multiplication or division, as shown in Figure 2. It is worth observing that the trigonometric quantity (sine, cosine or tangent) is the *multiplier* here, and so in each case we either multiply or divide by that. We *never* need to divide by the side length; indeed, this would produce nonsense dimensionally.

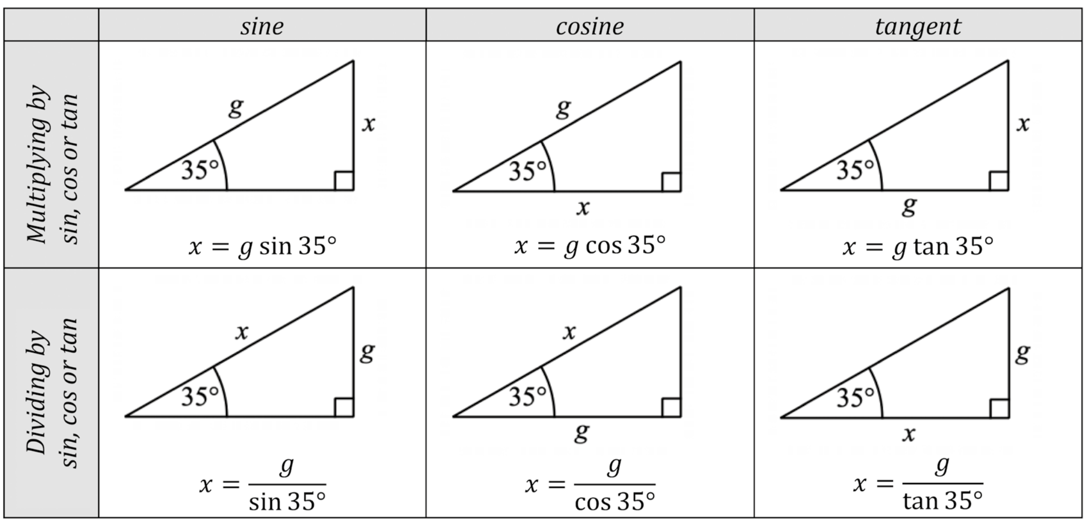


Figure 2. Solving the triangles in the usual way, using sine, cosine or tangent.

One teaching approach that avoids presenting 'SOHCAHTOA' is to give out a version of Figure 2 (without the column and row titles) alongside a set of missing-side questions and ask students to find for each question the appropriate match from Figure 2. Rather than the teacher explaining what they should do when, it becomes the students' task to figure out the pattern and be responsible for explaining. Then you can ask questions like "When do you divide?" or "When do you use cosine?" and students can explain and provide examples. As they struggle to put into words what they already informally understand, they end up using words like 'opposite side', and it is possible then to pick up on this kind of language and formalise it, with *rules following meaning* rather than the other way round. For this reason, I deliberately *wouldn't* label the six pictures (e.g., A-F) for easy reference, because I don't want students referring to them in the discussion as 'Type C'; I want them to struggle to have to say things like "the one where you divide by sine", etc.

This is a teaching approach that begins with quite a lot of complexity, with everything thrown at students at once, and they tend to notice things like $\cos 55° = \sin 35°$. But there are various ways to build up in a more stepwise fashion. One is to keep the angle fixed until everything else has been varied, and I quite like 35° as an angle for this, partly because tan 35° is very close to 0.7.

Just using sine

However, the teacher I mentioned at the start was arguing that there is far more complexity here than is really needed if all we want to do is be able to solve right-angled triangles efficiently. Consider the two problems shown in Figure 3. In a sense, they are two problems, but in another sense they are the same problem. Traditionally we would use sine to solve the left one and cosine to solve the right one, but might it not be easier, the teacher argued, to use sine (say) for both, and just calculate 90 – 55 = 35 in the second case to find the appropriate angle? Constantly switching between sine and cosine, and opposite and adjacent, the teacher argued, leads to numerous mistakes, whereas calculating complementary angles is easy. Keep it simple, and just teach sine!

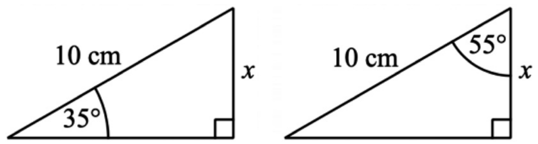


Figure 3. Two different problems or the same problem?

It is clear that 'cosine questions' can always be easily answered in this way by using sine. But what about 'tangent questions'? The other teachers present thought that this was a major problem with the 'just use sine' approach, but the teacher merely conceded that 'tangent questions' involve *two* steps. However, both of these steps are still 'the same kind of thing', so nothing new needs to be learned: we just use sine twice. "I would much rather do one thing twice than do two different things," he said. "Students just get twice as much practice when they hit a 'tangent question'!" Using sine always requires the hypotenuse, so if your question neither has nor wants the hypotenuse, then you start by working it out. Then your question reduces to one you already know how to do. Figure 4 shows how this teacher's 'just use sine' approach works in the six possible scenarios.

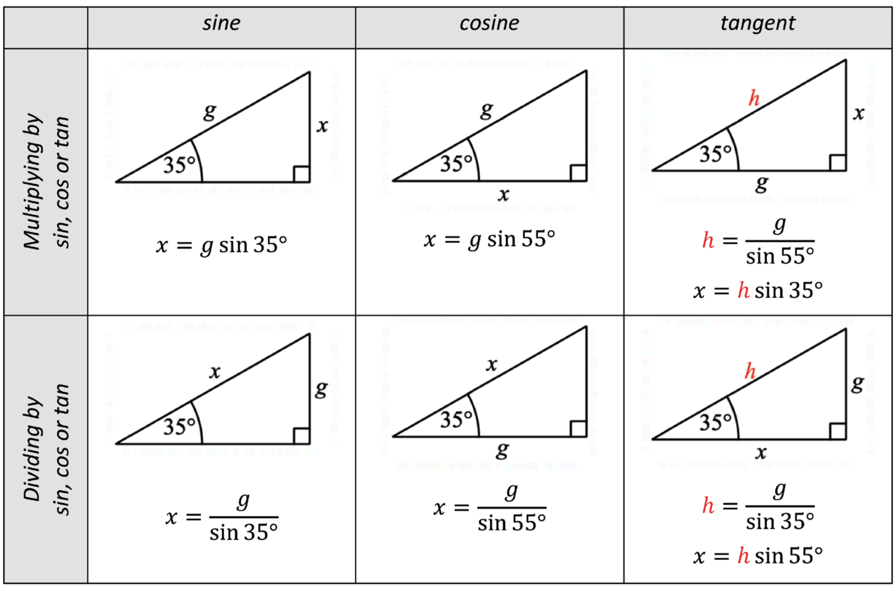


Figure 4. Solving the triangles using just sine. (Added unknowns in red.)

Finding angles is also relatively unproblematic with 'just use sine'. Figure 5 shows this, in comparison to the usual method. As far as this teacher was concerned, there is nothing to remember here. You always use sine, and if there isn't a hypotenuse then you work that out first, possibly by using Pythagoras's Theorem, which is useful revision, and it is always the 'easy' case of finding a hypotenuse, never a leg.

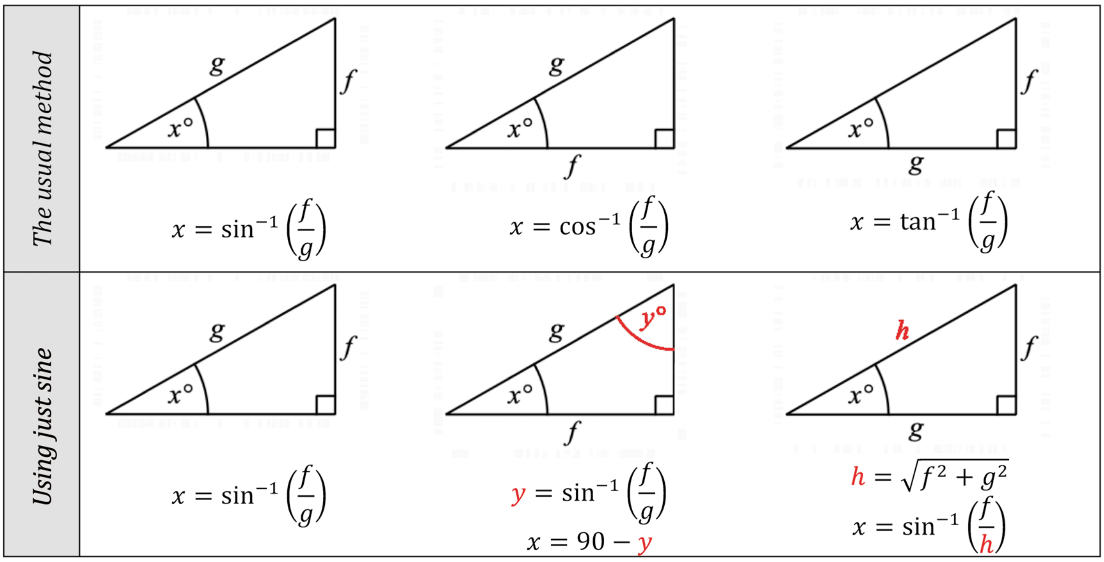


Figure 5. Finding an angle, given two sides. (Added unknowns in red.)

Just using the *sine rule*

In fact, right at the beginning of this conversation, when the teacher said that he 'just uses sine', I initially misunderstood what he meant. I thought that he meant that he just uses the *sine rule*, as the more general way of solving triangles (including non-right-angled ones). Why teach 'SOHCAHTOA' for the special case of right-angled triangles, when you have to go on to teach sine rule and cosine rule for solving general triangles, and those more general methods will work in all cases, including in the simple cases when the triangle is right-angled?

The cases with 'tangent' in Figure 4 can be simplified to:

$$x = \frac{g \sin 35°}{\sin 55°} \qquad \text{and} \qquad x = \frac{g \sin 55°}{\sin 35°},$$

and these formulae are very suggestive of the sine rule, which they are equivalent to.

The sine rule can be viewed as the relation that, in any triangle, $a \propto \sin A$ or, equivalently, that $\frac{a}{\sin A}$ (or $\frac{\sin A}{a}$) is a constant for any given triangle. The only convention needed to make sense of this is that an angle and the side *opposite* to that angle are given the same letter (upper case and lower case respectively). It is intuitively plausible to students that the longest side of a triangle will be opposite the largest angle, and the shortest side of a triangle will be opposite the smallest angle (Note 3).

The sine rule has an 'advanced' aura about it, and typically gets taught much later, and not necessarily to all students – and there is no doubt that the sine *rule* is more complicated than the sine *ratio*. But the relevant question is whether it is more complicated than *the total of the sine, cosine and tangent ratios*, with all of the switching about that moving among them entails? With the sine rule, it's always sine, so we don't have to decide whether it's sine or cosine or tangent. With the sine rule, we don't have to worry about labelling 'opposite' and 'adjacent' sides. There is more uniformity in the procedure when using the sine rule. Figures 6 and 7 show how the different possibilities work out when only using the sine rule.

Although these look complicated, they all follow the same

$$\frac{\blacksquare}{\blacksquare} = \frac{\blacksquare}{\blacksquare}$$

pattern, and sin 90° always immediately simplifies to 1, and so they are really just two-step processes. Additionally, you never need to multiply up by the unknown, so the rearrangings are always of the most straightforward kind. In a sense, of course, using the sine rule in a right-angled triangle is overkill – a 'sledgehammer to crack a nut', like using the quadratic formula to solve $x^2 = 36$. But, on the other side of the scales, you can now solve lots of non-right-angled triangles too 'for free', and every question becomes essentially the same.

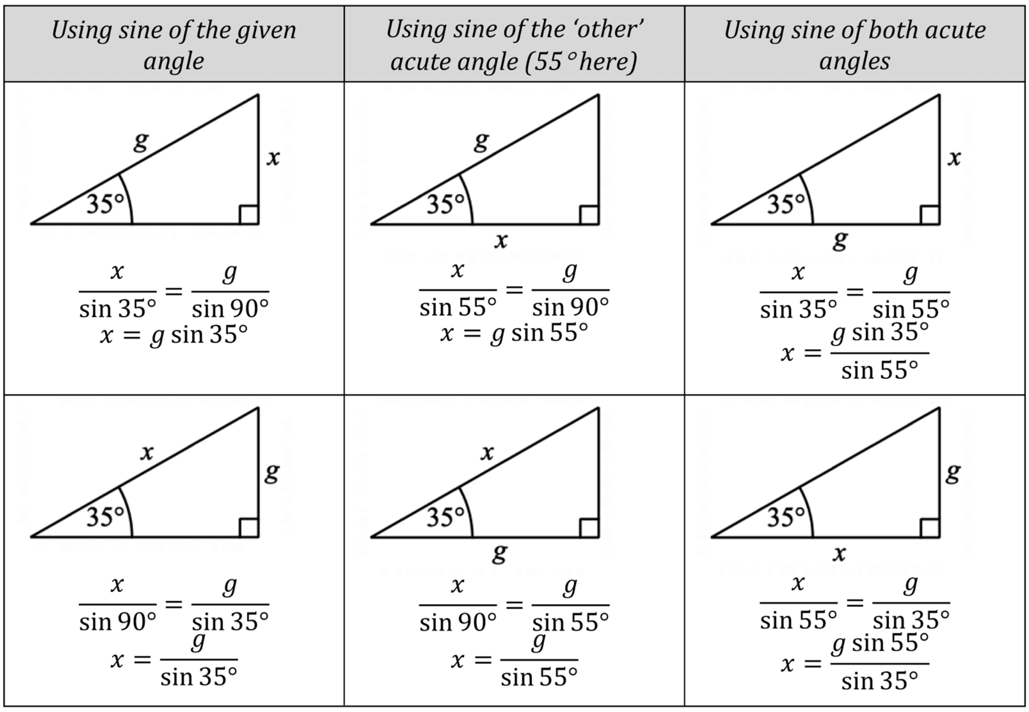


Figure 6. Solving the triangles using the sine rule every time.

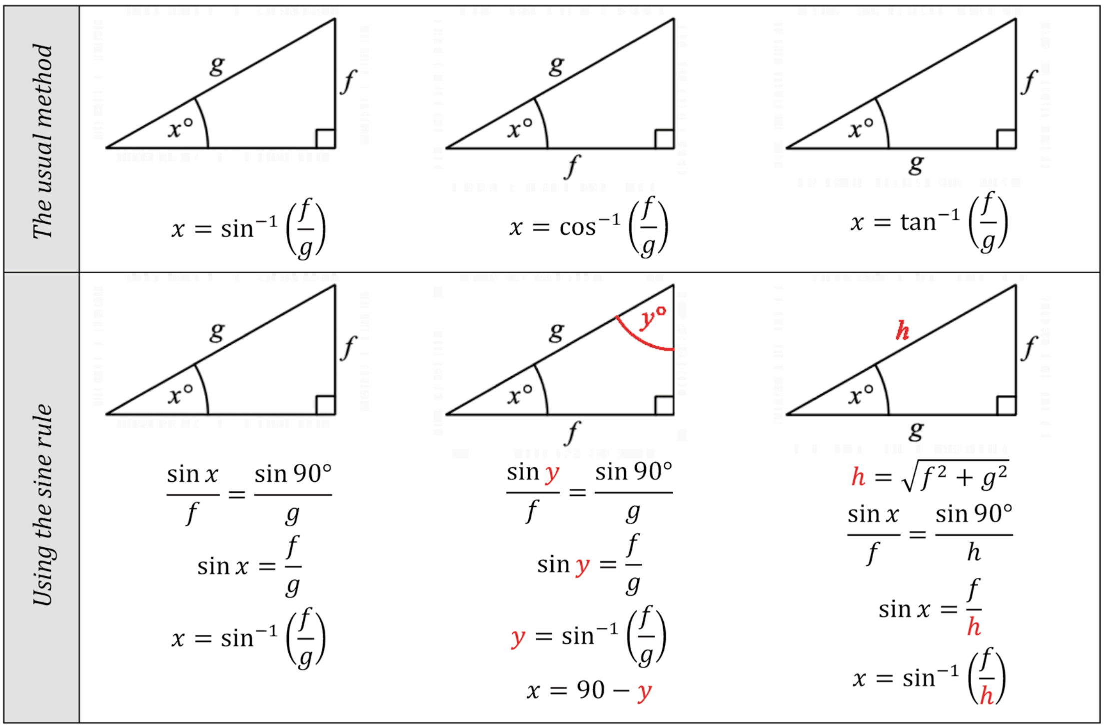


Figure 7. Finding an angle, given two sides, with added unknowns in red.

Conclusion

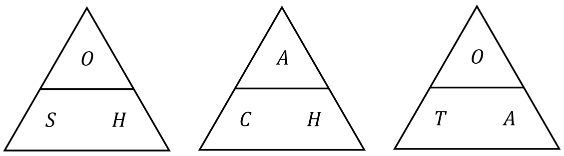
This is very much a thought experiment. I am not advocating either 'just use sine' or even 'just use the sine rule'. How would you prove the sine rule without already having the definition of sine? Students do need to know cosine and tangent as well, and their graphs, as they are important functions in their own rights. Indeed, rather than any of these methods, I would prefer to take a unit-circle kind of approach (see Hewitt, 2007), and I think that there are lots of advantages to this (see Foster, 2021).

However, I enjoyed thinking about these possibilities stimulated by the teacher's remark, because I think there is very often in mathematics teaching a trade-off between having lots of different methods and one more general method that is a bit harder or longer. The more general method seems like overkill in many of the commonly-encountered situations, but it has the virtue of consistency. In general, I think it is tricky to weigh up the pros and cons of which way is better.

Notes

1. Many readers will be familiar with SOHCAHTOA as a commonly-used mnemonic that gives the definitions of sine, cosine and tangent in a right-angled triangle in terms of sides that are opposite or adjacent to the angle of interest: Sine is Opposite over Hypotenuse; Cosine is Adjacent over Hypotenuse; Tangent is Opposite over Adjacent.

2. In some versions of SOHCAHTOA, formula triangles (Foster, 2021) are incorporated into the mnemonic as:



   This means that students can bypass the rearranging of equations step and write down the formula in the appropriately rearranged form as the first line of their solution.

3. A good way to convince students intuitively of this is to pose it as a conjecture and ask whether it is always, sometimes or never true. Students will naturally produce lots of sketches to try to 'break' it and find a counterexample. Note that this statement assumes a *scalene* triangle, which has a 'largest' and a 'smallest' side. Some isosceles triangles have only a largest or a smallest side, and some (isosceles triangles that are also equilateral) have neither.

References

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Keywords: Efficiency, Methods, Trigonometry, Sine rule, Rearranging equations, SOHCAHTOA.

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