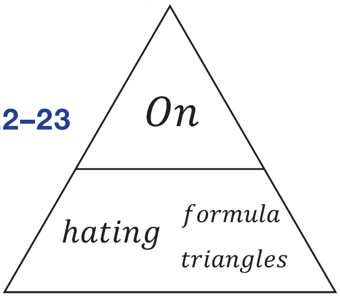
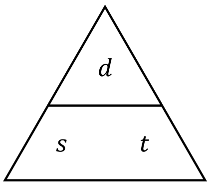
Colin Foster
ELECTED MA PRESIDENT FOR 2022–23

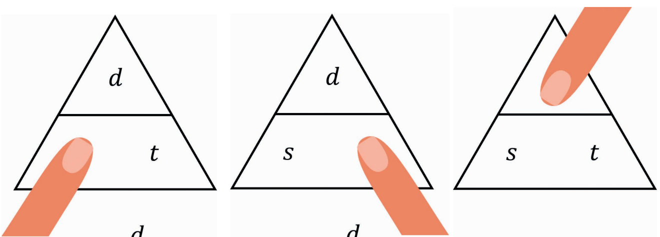


Everyone loves to hate formula triangles (e.g. Beveridge, 2014; Koenig, 2015). If your *raison d'être* for teaching mathematics is 'teaching for understanding', and everything you do in the classroom is geared towards sensemaking, then formula triangles feel like the antithesis of this. They are procedural, gimmicky and error-prone. Other writers are less absolute in their condemnation, but express strong reservations. Mason (1999) describes formula triangles as a mnemonic, and cautions that "there is nothing more useless than a mnemonic you cannot deconstruct, or a mnemonic that does not come to mind when it is needed" (p. 195). Southall (2016, p. 52) concedes that a formula triangle "can be a useful method once fully understood and recognised as a shortcut. But it should not be derived from less efficient manipulation of algebra and it should not be the first port of call for teachers to help students get to an answer."

Formula triangles crop up in various mathematics topics: compound measures, circle circumference and area, trigonometry – anywhere where there is a $y = mx$ proportional relationship – and all over the science curriculum. For example, for calculating speed-distance-time, a formula triangle could be used with s for speed, d for distance and t for time [Note 1].



This may be unnecessary, but, just for any reader who might not be familiar with them, the idea is to cover up the letter that you want and then read off the appropriate rearranged formula:



$s = \frac{d}{t}$ $t = \frac{d}{s}$ $d = st$

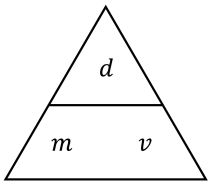
It is frequently reported that students misuse these triangles; for example, by not appreciating that the lower two letters need to be *multiplied* together, or by covering, say, the t but still reading off "distance multiplied by speed" instead of "distance divided by speed". There is also the problem of getting the letters in the right positions before you start. The two lower letters can be either way round, but we have to get the right letter at the top. I recently saw a lesson on 'compound measures', which was following on from a lesson on 'speed-distance-time' in which the formula triangle above had been used. It was not really clear to me how this lesson 'followed on', other than density-mass-volume being another $y = mx$ type of relationship with some connection to science. The teacher began by saying that this is another topic where we can use a formula triangle, and he drew a blank triangle. He then said:

"Density..." [writing d at the top]

"...equals mass..." [writing m on the bottom left]

"...divided by..." [he might have said "timesed by" – I am not sure]

"...volume." [writing v on the bottom right]



So, this is the wrong formula triangle for density, because mass should be on the top. But it is an easy mistake to make, especially when standing at the front of the class, and perhaps exacerbated by the fact that "d" for distance was on the top of the speed-distance-time one in the previous lesson [Note 2]. It seemed that the fact that this triangle is all about density, and "density" is the first thing in the formula that defines it, made it tempting to write the d first, in the principal position. If a teacher could do this, then surely this is something that a student might also do.

This error soon got picked up, but not until after the students had done some incorrect practice with this faulty triangle, and began to notice that the answers were not coming out right. More attention to the correct units, as well as to what density *means*, might have highlighted the error sooner. This made me reflect that this lesson wasn't really in any sense about density, because there was no thinking about how two objects of different masses and different volumes could have the same density or how two objects of the same mass or volume could have different densities. Unless students see in what sense 4 kg taking up 20 cm^3 of volume is 'the same' as 40 kg taking up 200 cm^3 of volume – and different from 4 kg taking up, say, 10 cm^3 of volume – then it is hard to claim that this lesson is about much more than just shuffling numbers around. The teacher reflected afterwards that during the time spent with the 'wrong' formula triangle the students were 'at least practising using the triangle', which I suppose is true. I guess we could even call d the momentum and say that it was '*m*ass × *v*elocity', and they were doing correct calculations in that context.

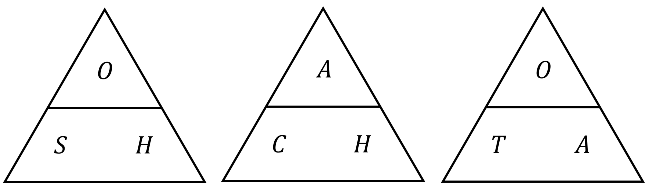
However, could it be that, despite all these problems, formula triangles do have something going for them? After all, I really *do* want students to be able to switch smoothly between the three equivalent equations $s=\frac{d}{t}$, $d = st$, and $t=\frac{d}{s}$ in this kind of relationship, and quickly 'see' the other two whenever they see any one of these. It may be tempting to just insist that they should do this by rearranging the equations 'properly' every time, by performing the same operations on both sides. But is this not perhaps an example of the 'curse of knowledge'? As an expert, I might say that I don't need crutches like formula triangles, so students shouldn't either (Foster, 2019). But it is possible for experts to underestimate the difficulty of 'doing things the long way'. While it might be realistic for students to rearrange step by step if they are doing this in isolation, if this is needed as part of some bigger task, such as finding the hypotenuse of a right-angled triangle, given an angle and the adjacent side, and the students are having to think, "Is it Pythagoras? Is it trigonometry?" "If it's trigonometry, is it sin, cos or tan?" and so on, by the time they, hopefully, correctly write down $\cos 30° = \frac{6}{x}$ if they still have two steps from this to get, via

$$x \cos 30° = 6$$

to

$$x = \frac{6}{\cos 30°},$$

then they may be suffering from cognitive overload before they get to the end (Foster, 2019). If every line of working has a 90% chance of being correct, then the probability of getting the correct answer at the end of n steps is 0.9^n, which is essentially zero for n greater than a few. On the other hand, a student who remembers "SOHCAHTOA" in this formula-triangle form [Note 3]



can then, by covering the H in the second triangle, immediately write down

$$H = \frac{A}{C}$$

and then perhaps get straight to

$$x = \frac{6}{\cos 30°}$$

with minimal fuss. But the question is whether this easier-ness comes at the cost of understanding.

One difficulty with rearranging these equations is that, in a sense, the key relationship is the *product* $y = mx$, because both $\frac{y}{m} = x$ and $\frac{y}{x} = m$ can be obtained from this in one step. So, rather than seeing the key fact as the definition of speed as a rate:

$$s = \frac{d}{t}$$

suppose that instead we saw speed as the *scaling factor*, the constant of proportionality between time and distance. Speed is the conversion factor that tells you how much distance you get for each unit of time. This would allow us to see $d = st$ as the starting point, matching the general pattern of $y = mx$ for straight-line graphs through the origin.

Then we have:

$$d = st$$

$$\frac{d}{s} = t \qquad\qquad \frac{d}{t} = s$$

Here we find the time by dividing the distance travelled by the scaling factor, the speed.	Here we find the scaling factor, speed, by seeing how much distance we get for each unit of time.

Some formulae are commonly presented in their product forms anyway, such as $V = IR$ for voltage, current and resistance, or $F = ma$ for Newton's second law. But others are generally presented as quotients, especially where the 'new' quantity, like 'speed' or 'density', is defined as a rate. Does this matter? Would it perhaps make teaching density calculations easier, for instance, if we started by saying that the key fact is that we have a nice way of thinking about the *mass* of an object: the total mass is equal to the amount of mass you get in every cubic centimetre (the density) multiplied by the number of cubic centimetres (the volume). And, from there, we get the other two formulae:

$$m = dv$$

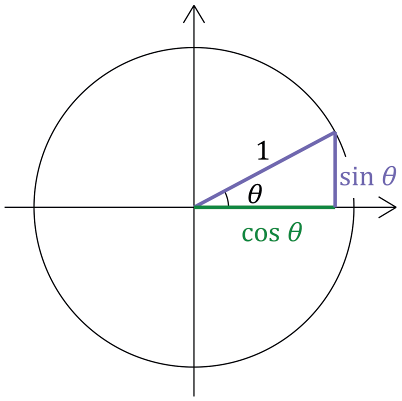
$$\frac{m}{d} = v \qquad\qquad \frac{m}{v} = d$$

Here we find the volume by dividing the mass by the scaling factor, in this case the density.	Here we find the scaling factor, the density, by seeing how much mass we get in each unit of volume.

$opp = hyp\ sin$		$adj = hyp\ cos$		$opp = adj\ tan$	
$\frac{opp}{hyp} = sin$	$\frac{opp}{sin} = hyp$	$\frac{adj}{hyp} = cos$	$\frac{adj}{cos} = hyp$	$\frac{opp}{adj} = tan$	$\frac{opp}{tan} = adj$

Operating in this way with trigonometry means starting with the three formulae at the top in the figure above [Note 4].

This approach perhaps also has a natural link to the way of beginning teaching trigonometry which starts with the unit circle (see Hewitt, 2007), rather than with ratios in right-angled triangles. In the unit-circle approach, sin θ is defined as the height of a right-angled triangle with angle θ at the centre of a unit circle, and cos θ is defined as the base of the same triangle (Figure 1). We find vertical lengths in *non*-unit circles (where $hyp \neq 1$) by multiplying sin θ by the scaling factor *hyp*, which is whatever the radius of the circle is, and similarly for cos θ. The unit-circle approach very naturally extends to angles greater than 90° [Note 5], which will need to be encountered later if students continue with mathematics, so perhaps it is an efficiency to begin this way.



So, I am all for abandoning reliance on formula triangles, but not if that just means burdening students with cumbersome formula rearrangements that tax their working memory and leave them more prone to making errors. However, if we prioritise the product form of common $y = mx$ kinds of linear relationships, might we potentially make rearranging these equations easier, while building stronger connections to multiplicative relationships more broadly, with their associated graphical understandings?

Notes

1. Sometimes the words 'speed', 'distance' and 'time' are used rather than letters.

2. The position of the *s* for speed in the speed-distance-time triangle can be further confused with *s* for displacement in the velocity-displacement-time version of this triangle.

3. Except for the student who told me at the end of a lesson that he found it easy to remember SOHCAHTOA because it has CAT in the middle, and he likes cats. He was halfway down the corridor before I realised that SOHCAHTOA *doesn't* have CAT in the middle!

4. I have written these with the trigonometric function *second* (e.g., *opp* = *hyp sin* rather than *opp* = *sin hyp*) partly to avoid the calculator error of working out sin(30° × 6) when 6 sin 30° is required, but mainly because it can then be seen as being in the form $y = mx$, where the hypotenuse functions as m, the scaling factor: this is the number of times that the unit circle triangle needs to be scaled up to make the required triangle.

5. The function tan θ has to be treated differently.

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