Proportionality with Malcolm

by **Colin Foster**

I came to the University of Nottingham in 2012 for one reason only: Malcolm Swan. I had known Malcolm for many years, mainly through the Institute of Mathematics Pedagogy that Anne Watson and John Mason organize each summer at Oxford. At Nottingham, I had the great privilege of working with Malcolm on design research projects, in particular the Mathematics Assessment Project (**http://map.mathshell.org/index.php**). The design of those 100 formative assessment lessons was truly collaborative, and it was a joy to work with Malcolm alongside Clare Dawson, Sheila Evans and Marie Joubert. Malcolm had the ability to raise a draft task to the next level. I would turn up to a meeting with what I thought was a pretty good task, and Malcolm would say something nice about it – and then, with a few deft moves, transform it into something vastly greater! It was a wonderful experience to be part of, and it was just as enjoyable to watch him do the same thing to other people's tasks – and I became fascinated by how he facilitated this process of collaborative lesson design.

Malcolm was an excellent mathematician, and was always interested in the underlying mathematics behind a task, as well as any tangential mathematics that could come out of working on it. I remember several occasions where Malcolm used the slide shown in Figure 1, asking "Which statements *define* proportion?".

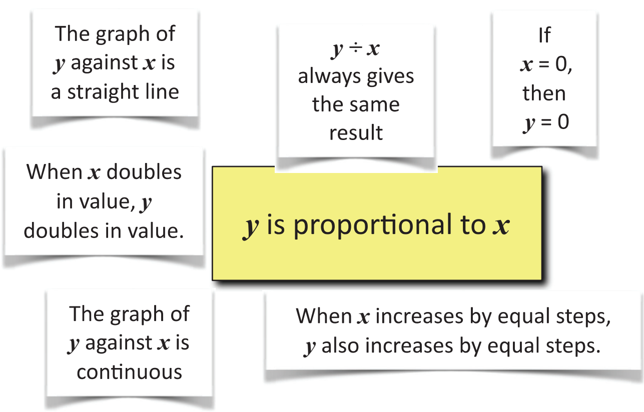

Fig. 1 Which statements *define* proportion?

At first glance, it looks like a task you could give to secondary-age pupils, but, like everything from Malcolm, it has hidden depths! In particular, the pair of statements:

- when x doubles in value, y doubles in value, and
- the graph of y against x is continuous

always provoked much discussion. Are these two, taken together, sufficient to determine that y is proportional to x?

The first statement on its own is often used in classrooms to explain to pupils what direct proportion is. But there is nothing special about doubling here. More generally, what is meant is that if x is scaled up m times then y is also scaled up m times, *and that this is true for any m*. This is quite a complicated statement! Algebraically, $y \propto x$ is equivalent to $y = kx$, where k is a constant. So if x suddenly becomes mx, then y must become $k(mx) = m(kx) = my$. So, multiplying x by any constant will multiply y by the same constant. That is the essence of proportionality, as understood algebraically. But the two constants, k and m, make this quite a complicated idea to express in words, so it is perhaps understandable that 'doubling' is sometimes used to stand for all possible scalings.

It seems natural to think that this 'if x doubles then y doubles' rule must produce a straight line through the origin, but this is to confuse a statement with its converse. If y is proportional to x, then it follows that if x doubles then y doubles. But the converse turns out *not* to be true. There is some nice mathematics here. It is tempting to think that the functional equation $f(mx) = mf(x)$ implies that f is a linear function, but in fact the definition of linearity has *two* parts to it:

	homogeneity:	$f(mx) = mf(x)$
but also	*additivity*:	$f(x_1) + f(x_2) = f(x_1 + x_2)$.

Additivity ⇒ homogeneity, since, for example, if we let $x_1 = x_2$ then our additivity condition becomes $f(x_1) + f(x_1) = f(x_1 + x_1)$, or $2f(x_1) = f(2x_1)$. But homogeneity ⇏ additivity. This suggests that perhaps we will be able to find a function such that, whenever you double the x value, the y value doubles, but where y is *not* proportional to x? You might like to have a go before reading on.

One possibility is to go for something discontinuous. For example,

$$f(x) = \begin{cases} 2x, & x \in \{1, 2, 4, 8, 16, 32, \ldots\} \\ 3x, & x \in \{3, 6, 12, 24, 48, \ldots\}. \end{cases}$$

Here, when any valid x value is doubled, $f(x)$ also doubles, but it is clear from Figure 2 that we really have *two* series of points, one on the line $y = 2x$ and the other on the line $y = 3x$, and so it would be wrong to say that $f(x) \propto x$, since it depends which line of dots you are on (Note 1).

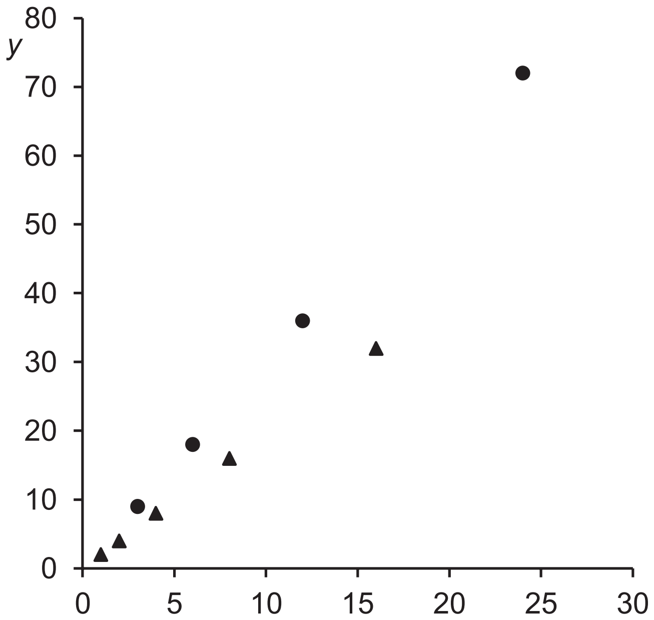

Fig. 2 A discontinuous counterexample

Well, OK, we have done it, but this is a very contrived and inelegant example – and quite unlike the kinds of functions pupils are used to meeting in school. What we really want is a *continuous* example (Malcolm's second criterion), where the functional equation $f(2x) = 2f(x)$ is satisfied but $y \neq kx$.

Of course, Malcolm knew all about the mathematics of this, but he had a lovely way of always letting other people score the goals! When he presented at a conference, it was a delegate (Note 2) who posed the equation:

$y = x\sin(2\pi\log_2|x|)$.

We can see here that replacing x by $2x$ gives

$$\begin{aligned} 2x\sin(2\pi\log_2|2x|) &= 2x\sin(2\pi(\log_2 2 + \log_2|x|) \\ &= 2x\sin(2\pi(1 + \log_2|x|) \\ &= 2x\sin(2\pi + 2\pi\log_2|x|) \\ &= 2x\sin(2\pi\log_2|x|) = 2y. \end{aligned}$$

The logarithm to base 2 (multiplied by 2π) is a neat way to convert multiplication by 2 into the addition of 2π, which (since this is the period of the sine function) has no effect on the value of the sine, meaning that it is only the 2 at the front of the equation which remains. Looking at the graph (shown in Figure 3), we can see how any straight line through the origin that intersects the curve at an x-value of x_1, will intersect it again at $2x_1$, $4x_1$, $8x_1$, and so on. Clearly, it would be possible to make this kind of solution work for any multiplier (e.g. 3 instead of 2), just by using a logarithm to a different base (3 in this case). But we cannot make a *single* formula like this which will work for *more than one* multiplier.

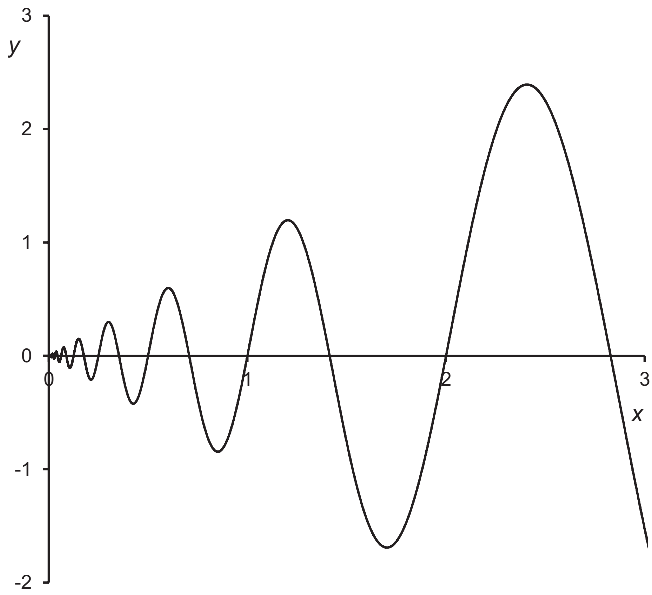

Fig. 3 The graph $y = x\sin(2\pi log_2|x|)$

Malcolm had a wonderful way of digging into ostensibly elementary mathematics and bringing out something at just the right level for teachers to appreciate something new. He constantly placed the teacher in the role of learner in order to allow them a better understanding of pupils' experiences. And he just loved working with people on mathematics.

Notes

1. My former colleague, John Cooper, pointed out that a nicer solution along these lines would be: $f(x) = x$ if $x \in \mathbb{Q}$; $f(x) = 2x$ if $x \notin \mathbb{Q}$.
2. I am grateful to Professor David Jabon at DePaul University for pointing me to **http://eqworld.ipmnet.ru/en/solutions/fe/fe1111.pdf**

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