Questions pupils ask: How can probability density be greater than 1?

By **Colin Foster**

When Normal distribution curves are plotted, the vertical axis is often omitted. All that matters with a continuous probability distribution is the *area under the curve* between particular values of the random variable that is plotted on the horizontal axis. It is area under the curve that corresponds to probability, rather than the values on the vertical axis. But sometimes computer software or books do show the probability density values on the vertical axis, as in Figure 1, and this can lead to questions from students.

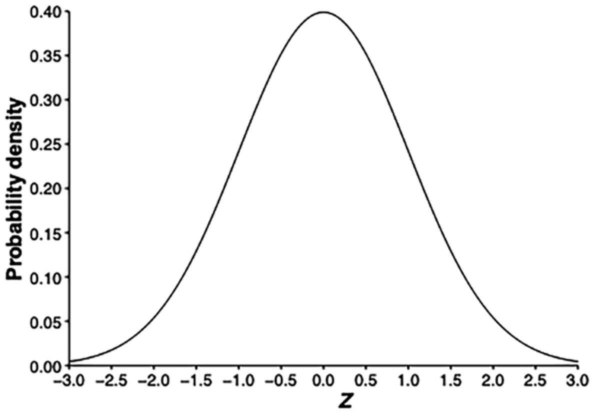

Fig. 1: Standard Normal distribution, $Z \sim \mathrm{N}(0, 1)$

In cases such as Figure 1, where the probability density values are all less than 1, showing the vertical axis may not lead to any remarks. But this may be because students are unconsciously treating these probability density values as though they are probabilities. This can mean that when students see other Normal curves, where the probability densities do exceed 1, they may become confused.

In general, probability density values can easily exceed 1. For example, if we reduce the standard deviation by a factor of 4 (i.e. to ¼, or 0.25), that will make the Normal curve narrower, which necessarily also makes it taller, since the total area under the entire curve must remain 1, otherwise it wouldn't be a probability distribution. The result, in Figure 2, is a curve with probability density values 4 times as large, and many of these values are now greater than 1, as shown by the portion of the curve that is above the horizontal dashed line. Students may feel that something must be wrong here, because probability density values greater than 1 seem impossible.

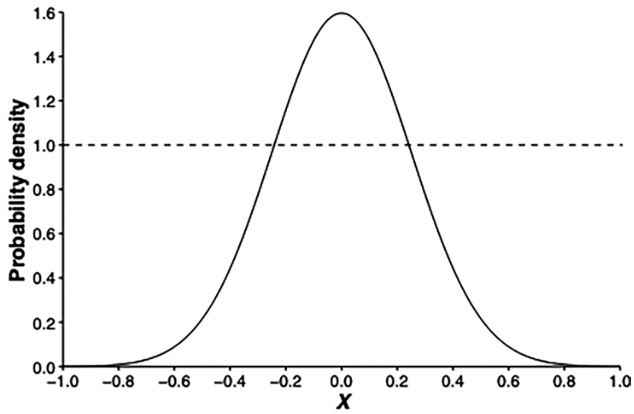

Fig. 2: Normal distribution, $X \sim \mathrm{N}(0, 0.25^2)$

One way to explore this is to play with the probability density formula for the Normal distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

This is a formula which is sometimes *shown* to students, but they never get to do anything with it, because it can't be integrated analytically, so they have to rely on technology (or old-fashioned tables) to find any probability values. But here is an opportunity to see what happens when you substitute numbers into it. If we focus on the peak, where $x - \mu = 0$, then the $e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ factor reduces to 1, and we obtain $\frac{1}{\sigma\sqrt{2\pi}}$. This means that the standard Normal distribution (where $\sigma = 1$) has a peak probability density of $\frac{1}{\sqrt{2\pi}}$, which is 0.399 (correct to 3 decimal places). This corresponds to what we see in Figure 1. For Normal curves in general, with standard deviation σ, the peak probability density $\frac{1}{\sigma\sqrt{2\pi}}$ is inversely proportional to σ. For example, if $\sigma = 0.25$, we obtain $\frac{4}{\sqrt{2\pi}}$, which is 1.596 (correct to 3 decimal places), which shows why the peak in Figure 2 is 4 times the height of that in Figure 1. Indeed, if we wanted the peak of a Normal curve to have a height of, say, 10, we would just need a standard deviation of $\frac{1}{10\sqrt{2\pi}} = 0.040$ (correct to 3 decimal places).

Of course, we don't really need to use the formula to see any of this. Changing the standard deviation from 1 to σ means stretching the curve horizontally by a factor of σ.

Since the area under the curve must continue to be 1, this means simultaneously stretching the curve vertically by a factor of $\frac{1}{\sigma}$. We can make a Normal curve as tall as we wish by making σ as small as we like. Whenever we look at any portion of the *area* under any of these curves, that is always guaranteed to be less than 1, and this is what matters, because it is area that corresponds to probability.

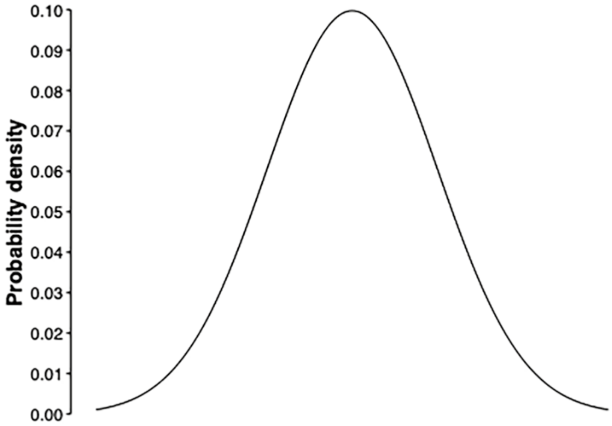

Fig. 3: The standard deviation of this Normal curve must be about 4

Indeed, we can actually determine the standard deviation from the peak value of a Normal curve without needing any knowledge of the values on the horizontal axis. For example, in Figure 3, with a peak probability density of 0.1, and no horizontal axis information, although we can't tell what the mean is, we know that the standard deviation must be equal to $\frac{10}{\sqrt{2\pi}}$, or about 4.

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DOMINO PROBLEM 6: SOLUTION

By **Chris Pritchard**

For Domino Problem 6, let the common radius have length r and the domino be 1 by 2. Then the green triangle has sides

$$1, r \text{ and } \frac{1}{2}(r+1).$$

By Pythagoras' Theorem,

$$r^2 = 1 + \frac{1}{4}(r+1)^2$$

$$4r^2 = 4 + r^2 + 2r + 1$$

$$3r^2 - 2r - 5 = 0$$

$$(3r-5)(r+1) = 0.$$

And so, with $r = -1$ inadmissible, $r = 5/3$.