# Questions pupils ask: Is infinity a number?

By **Colin Foster**

Every mathematics teacher has to arrive at work prepared for infinity. According to Hermann Weyl, “Mathematics is the science of the infinite”, and Caleb Gattegno memorably said that mathematics is “shot through with infinity”. I certainly find that some reference to infinity is quite a common occurrence in school mathematics lessons.

When working on powers of 10, I would often begin by asking students to tell me ‘a large number’ or ‘the largest number you know’. They would offer words like ‘million’, ‘billion’, ‘trillion’ – and even ‘zillion’ and ‘gazillion’. Occasionally, someone has heard of a ‘googol’, which is $10^{100}$, as opposed to the multinational corporation *Google*. When answers dried up, I would ask, “Anything bigger?” and someone would say “A trillion plus one” or “Two trillion” or “A trillion trillion trillion”! It’s a natural step from these sorts of responses to a perfectly valid proof that no largest number can exist. We suppose that there is one, and then we add 1 to it, and suddenly we have a larger one. This means that we were mistaken to think that there could have been a largest number (see Note).

I would generally write all of the various words on the board as they were offered, and afterwards we’d sort them out. ‘Zillion’ and ‘gazillion’ aren’t actually numbers – they just mean ‘a lot’. ‘Million’, ‘billion’, ‘trillion’ make a geometric sequence, multiplying by 1000, or adding three zeroes, each time. But at some point during this, sometimes quite near the beginning, someone is likely to offer an answer of ‘infinity’. And I think you really need to plan for that. If it catches you out, it’s hard to respond well.

What do you do with an answer of ‘infinity’? Of course, you can write the word on the board, along with the others, and stall for time. But then what? *Is* infinity a number? Does it belong in this discussion? Infinity seems unbeatable – the ‘add 1’ trick doesn’t defeat it, because ‘infinity plus one’ is (presumably) still infinity. We could write that kind of thing as an equation, perhaps introducing the infinity symbol:

$$\infty + 1 = \infty.$$

But then it certainly looks as though we’re treating infinity as a number – or, at least, as a very number-like thing. If you can add 1 to something, then surely that something must be a number? And yet if it were an ordinary real number then we would be able to subtract infinity from both sides, to obtain

$$1 = 0$$

And that can’t be right. So, do we just say that that happens to be ‘not allowed’ with infinity? Or do we say that infinity isn’t a number in the first place, so we should never have expected this to work?

My favourite way to talk about infinity is to use David Hilbert's story of ‘Hotel Infinity’ (Tahta & Hemmings, 2017), which has a countably infinite number of rooms, numbered 1, 2, 3, … At the start of the story, every room is occupied. However, Hotel Infinity can accommodate a new arrival if everyone shifts to the next room, that is one number higher, because that leaves room 1 vacant. Hotel Infinity was full before, and it’s still full now that the new guest has been accommodated. Hotel Infinity can even accommodate a countably infinite number of new guests, who perhaps arrive on ‘Coach Infinity’, because the guest in room $n$ can move to room $2n$, leaving countably infinitely many odd rooms for the new guests to take.

All of this is great fun – and mind-bending. But what is it showing? Is it telling us that infinity *isn’t a number*? Or that infinity is a number, but just of a very different kind from the real numbers that we’re used to? We might say that infinity isn’t a number, because it doesn’t have a location on the number line. Where would you put it? However far to the right you try to place it, it always needs to be to the right of that. We know that some things aren’t numbers, like a triangle or a potato. We’ve seen that a zillion is just ‘a lot’, rather than any specific number. So, there’s no problem with some things not being numbers. But infinity doesn’t feel like those non-numbers. It’s surely more like a number than it’s like anything else. Can’t we just say that it’s an *infinite* – or *transfinite* – number?

As students meet an expanding number system, they are forced to become increasingly relaxed about what ‘a number’ can be. They start with the counting numbers, or positive integers, which everyone believes are numbers. But as soon as we include zero we’re being controversial, because we're including a number that we *can't* count – and that we also can't divide by. So, we can make a good case that zero isn't a number, at least by the standards of the counting numbers. And pupils will often query whether zero is actually a (proper) number (Foster, 2023).

But instead of throwing out zero, we ask them to expand their notion of number to include zero. And then subsequently we include negative numbers, which are less than nothing, which seems impossible. And we include rational non-integers, which aren't 'the number of' anything, but we still end up including them as 'numbers', because what we mean by 'number' grows to accommodate it – a bit like how Hotel Infinity accommodates new guests, although without growing!

Later on, irrational numbers and then imaginary numbers come along. Like infinity, imaginary numbers can't be represented on the (real) number line: the number *i* hovers 1 unit above zero (Foster, 2018). But we end up convincing ourselves to accept complex numbers as a whole plane of numbers, and we still think of them as numbers. So, saying that infinity can't be a number because it doesn't fit conveniently onto our real number line seems to lack imagination.

If by 'number' you mean 'counting number', then zero, negatives, and all of the rest aren't numbers. If by 'number' you mean 'real number', then non-reals like *i* aren't numbers. If by 'number' you mean 'finite number', then infinity isn't a number. But why stop at finite numbers?

Infinity isn't really an unfamiliar or 'advanced' concept (Tahta & Hemmings, 2017). Infinite processes that never end are all around us, and certainly pop up frequently in school mathematics.

- How far can you travel around a circle before you reach an end?
- When working on angles, ask pupils to stand up and face the front of the room. Then turn clockwise through 1000°. Where will they be facing? What other angles lead to this same position?
- What is the order of rotational symmetry of a circle?
- Pupils meet infinite decimals, of the repeating kind, such as $\frac{1}{3}$, corresponding to the never-ending process of dividing by 3 and obtaining a remainder of 1. And they also meet irrational decimals, such as $\sqrt{2}$ and π.

For me, it's much more natural to say yes, infinity is a number. But don't expect infinite numbers to be like finite ones!

## Note

Is that the end of the proof, or do we then take that newly-created number as our new largest number and run through the argument again?

## References


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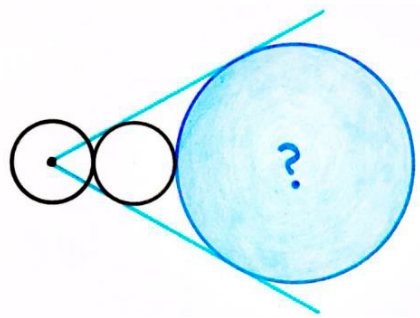
## Geometry Problem 26

### By Catriona Agg

The small circles each have area 1.

What's the area of the large circle?




Send your solutions to Chris Pritchard (chrispritchard2@aol.com) and we will publish the best.