Questions Pupils Ask: Why does a transformed graph move the wrong way?

By **Colin Foster**

I was interested to read John Below's article, 'Teaching why $y = f(x + a)$ translates the graph to the left' (Below, 2026), which followed up on Robin McLean's earlier article 'How do you teach the graph of $y = f(x + a)$?' (McLean, 2023). I certainly agree that this can be a real sticking point for students – hence the question, "Why does a transformed graph move the wrong way?" Ever since pupils encountered directed numbers, '+' corresponds to moving to the right and '−' corresponds to moving to the left. So, why is the graph of $y = f(x + a)$ a translation of the graph of $y = f(x)$ by a units to the *left*?

Prior to meeting this, pupils will have seen the transformations of $y = f(x)$ into $y = f(x) + c$ and $y = mf(x)$, where c and m are constants, and both of these graphs move exactly as expected. The '$+c$' shifts the y values *up* c units (down if c is negative) and the m scales the y values by a factor of m, stretching the graph vertically by a scale factor of m from the x-axis. The $f(x)$ values – the heights above the x-axis – behave perfectly reasonably. No surprises here.

The tricky cases

To address the two problematic transformations in the x direction, I prefer to use a graph that has nice integer coordinates but no actual formula to worry about. A function doesn't have to have an equation – it just needs to be some relationship between x and y in which every allowable x value is associated with exactly one y value. The graph of a suitable function is shown in Fig. 1.

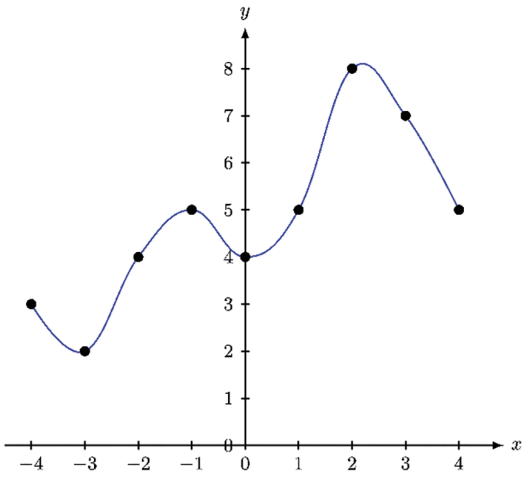


Fig. 1. The graph of a convenient function $y = f(x)$.

To make sense of notation like $f(x + a)$, pupils need to realise that, strictly speaking, $f(x)$ is not 'the function'. The letter f represents the function; $f(x)$ is the *output* of the function. For example, $f(1)$ is the output of the function f when the input is 1. I like to show the $f(x)$ values explicitly for convenient integer x values, as in Fig. 2.

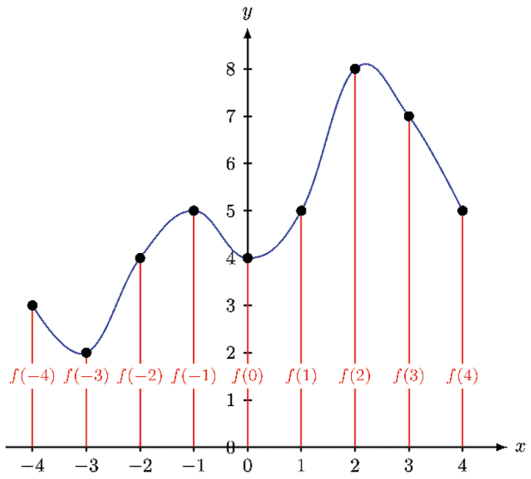


Fig. 2. Showing the $f(x)$ values for each integer x.

Now, if we translate the curve, and all its points, to the left by 1 unit, the y values have to move with it. This means that, for example, $f(3)$ will no longer be at $x = 3$; it will now be at $x = 2$, as shown in Fig. 3.

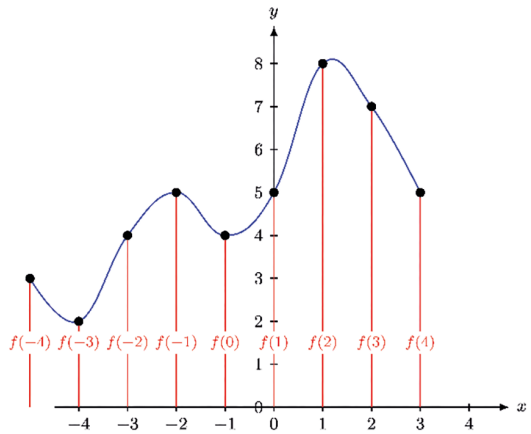


Fig. 3. Translating the graph 1 unit to the left to obtain $y = f(x + 1)$.

At every x value, we are now plotting not $f(x)$ but $f(x + 1)$. The x's are 'getting ahead of themselves', so to speak, by stealing the y value from the *next* x value along. For example, $x = 2$ is now associated with $f(3)$, not $f(2)$.

That means that our graph must now be $y = f(x + 1)$, not $y = f(x)$, and this corresponds to a shift of the graph 1 unit to the *left*.

In general, when we add a to x, we steal a y value from the x value that is a units to the right. This brings the graph from the right over to where we are, moving it a units to the left.

We can do something similar with transforming $y = f(x)$ into $y = f(mx)$. This time, each x value steals the y value from mx. For example, if $m = 2$, then when $x = 1$ we won't see a y value of $f(1)$ but of $f(2)$. And when $x = -2$, we won't see a y value of $f(-2)$ but of $f(-4)$. All of the y values get 'sucked in' towards the y-axis, as shown in Fig. 4. So, multiplying the x by m corresponds to stretching horizontally from the y-axis by a scale factor of $\frac{1}{m}$.

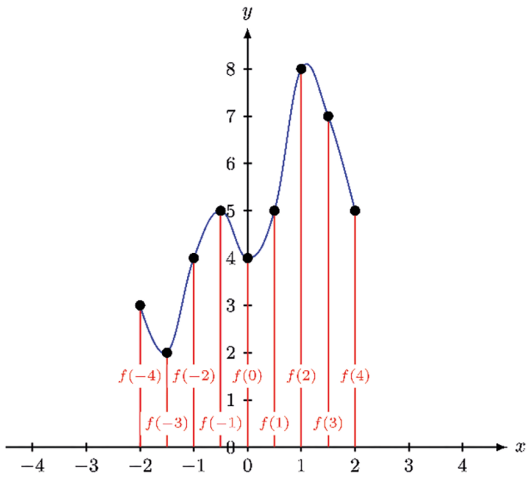


Fig. 4. Stretching the original graph horizontally by a scale factor of $\frac{1}{2}$ from the y-axis to obtain $y = f(2x)$.

Combining the tricky transformations

The most complicated and potentially confusing case is to ask what would happen if we *combined* these two transformations; in other words, how would we obtain a graph like $y = f(2x + 1)$?

We might think we should slide the graph in Fig. 4 to the left by 1 unit. But that would be a mistake. If we do that, as shown in Fig. 5, we can see that $f(4)$ ends up at $x = 1$, but when $x = 1$ the value of $2x + 1$ is 3, not 4. So Fig. 5 is actually showing us $y = f(2x + 2)$, not $y = f(2x + 1)$. The correct graph for $y = f(2x + 1)$ is shown in Fig. 6.

Why did this go wrong? The problem was that doing our stretch first has halved the separation of the red lines, so a subsequent translation of 1 moves us twice as far as we need to go. If we insist on doing the stretch first, then we must only translate to the left by $\frac{1}{2}$, not 1.

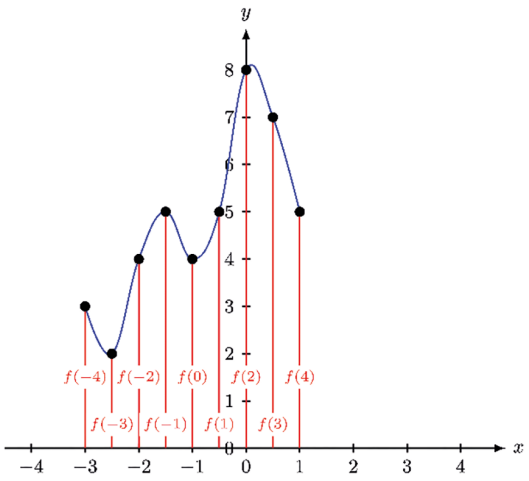


Fig. 5. Stretching the original graph horizontally by a scale factor of $\frac{1}{2}$ from the y-axis and translating by 1 unit left to obtain $y = f(2x + 2)$ not $y = f(2x + 1)$.

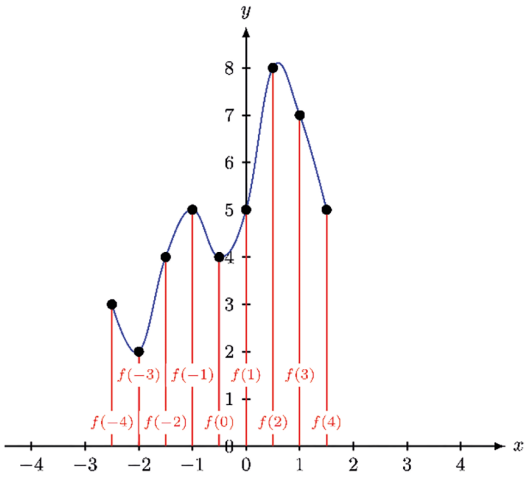


Fig. 6. Translating the original graph by 1 unit left and stretching horizontally by a scale factor of $\frac{1}{2}$ from the y-axis to obtain $y = f(2x + 1)$.

Alternatively, we can obtain Fig. 6 by first doing the translation to the left, by 1, and *then* stretching horizontally by a factor of $\frac{1}{2}$. This maintains $f(1)$ at $x = 0$, because no horizontal stretch from the y-axis will shift whatever the y value is at $x = 0$.

To students, 'doing the + 1 before the × 2' feels wrong; it is contrary to the priority of operations we normally follow. The confusion comes from thinking that the stretching transformation on Fig. 3 takes $y = f(x + 1)$ into $y = f(2x + 2)$. Instead, it takes $y = f(x + 1)$ into $y = f(2x + 1)$ (Fig. 6). We are not 'doubling everything inside the bracket', but simply switching x for $2x$. I think of this *not* as multiplication, but as 'change of variable', from x to $2x$.

This is easier to see in the algebra when the letter x appears more than once in the equation, such as in a quadratic like $y = (x - 1)(x - 3)$. To stretch its graph by a scale factor of $\frac{1}{2}$, parallel to the x-axis, we need $y = (2x - 1)(2x - 3)$. In expanded form, the original graph, $y = (x - 1)(x - 3) = x^2 - 4x + 3$, becomes

$y = (2x)^2 - 4 \times (2x) + 3$. Each x gets replaced by $2x$, but the constant terms are unaffected. That is what $f(x) \to f(2x)$ means.

Similarly, translating 1 to the left doesn't mean adding 1 to the contents of the bracket – it means surgically replacing every x with $x + 1$. This is why translating $y = f(2x)$ by 1 unit left led to $y = f(2x + 2)$, not $y = f(2x + 1)$. The translation leads to $y = f(2(x + 1))$, which is equal to $y = f(2x + 2)$.

Another way to think about Fig. 6 is to note that the y value at $x = -1$ is $f(-1)$, exactly as in the original graph $y = f(x)$ (Fig. 1). This happens because $2 \times (-1) + 1 = -1$. So, we could describe the transformation from $y = f(x)$ to $y = f(2x + 1)$ (Fig. 6) as a horizontal stretch with scale factor $\frac{1}{2}$, *but from the line* $x = -1$, rather than from the y-axis. We can think of this as being a *single* transformation – a one-way stretch – just from a different invariant line, rather than a combination of a stretch and a translation.

Perhaps different approaches, diagrams and explanations offer different insights. I suspect the ideal is for pupils to look at these transformations in several different ways. I wonder what other approaches readers use or prefer?

For an animated video version of this, please see https://www.twominutemath.org/gallery/horizontaltransforms.html.

References

Below, J. (2026). 'Teaching why $y = f(x + a)$ translates the graph to the left', *Mathematics in School* 55, 2 pp.18-21.

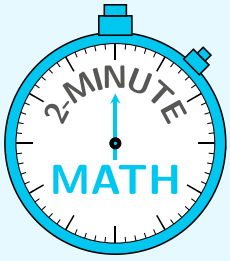
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Keywords: functions, graphs, transformations

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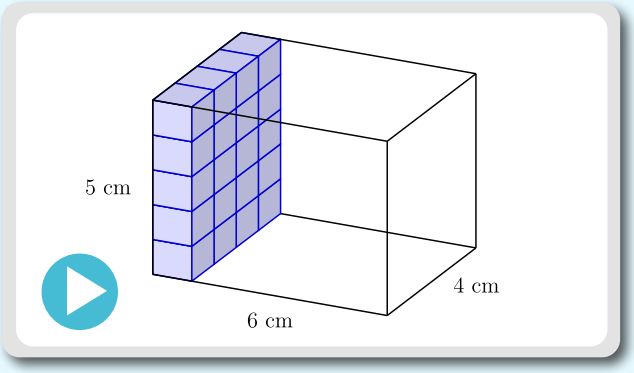
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