Reaching the 100th Square

by **Colin Foster**

I don't know where this problem originates (Note 1), but the idea is that you have a line or grid of 100 squares and an ordinary die. Each time you throw the die, you move forward that many squares. What is the probability that you will eventually land on the 100th square and not overshoot?

One way to begin is to simplify the problem to 10 squares so as to see what is involved. Since 6 is the largest number on an ordinary die, it is impossible to reach the 10th square in one throw, so at least 2 throws will be needed. On the other hand, even if we throw 1s every time, we still must reach the 10th square by the time we have thrown the die 10 times. This means that we need to consider the numbers of ways of reaching the 10th square in 2, 3, 4, 5, 6, 7, 8, 9 and 10 throws of the die. Table 1 shows the possibilities, and the total probability comes to $\frac{17\,492\,167}{60\,466\,176}$, which is 0.289 (correct to 3 significant figures). Did you expect the answer to be a bit less than $\frac{1}{3}$?

Now that we have done this, we can see that actually listing all of the ways of getting to the 10th square is unnecessarily laborious. This becomes clear if we think about a game with a small number of squares. For a 1-square game, the only way to win would be to throw a 1, so the probability would simply be $\frac{1}{6}$. For a 2-square game, the two ways of winning are to throw two 1s or a 2, so the probability would be $\frac{1}{6^2}+\frac{1}{6}=\frac{7}{36}$, which is 0.194, correct to 3 significant figures. In general, to get to the nth square in t throws, we need the number of partitions of n into t positive integers, all of which are less than 7. For $n < 7$ these are just the familiar numbers from Pascal's triangle shown in Table 2. For example, there are ${}^4C_2$ ways of getting to square 5 in 3 throws, because we have to choose 2 squares on which to rest in between our throws, and there are 4 possible squares we could rest on (since we can't rest on square 5, as getting there would be a win). So, in general, the number of ways of getting to the nth square in t throws will be ${}^{n-1}C_{t-1}$, provided that $n < 7$.

However, things become a little more complicated for $n > 6$, because this method will include partitions in which one of the throws is more than 6, which is impossible with an ordinary die. This alters the values for cells in which $t < n - 5$, which are shaded in Table 2. In each of these cases, we have to reduce the total number of ways by the number of ways that involve throwing a number greater than 6. This leads to an interesting jump in the graph of the final probabilities (Fig. 1).

Returning to the original problem with 100 squares, it would take far too long to tackle that by listing all of the possibilities. Simulation could be a good idea, and for the 10-square case, this is easy to perform in a spreadsheet – and it would be possible to extend this to the 100-square case. However, sometimes in probability the bigger problem is actually easier to solve than the smaller problem – at least if you are happy with an estimate – and that turns out to be the case here.

A very useful way to think about this problem is to consider the *expected score* when you throw an ordinary die. Since the values 1 to 6 appear with equal probability, the average score on each throw is $\frac{1+2+3+4+5+6}{6} = 3.5$. This means that, on average, we are moving in jumps of 3.5 squares, so this means that in the long run we will land on 'one in every 3.5' of the squares in the game. So, we should expect that the probability of landing on any particular very distant square will approach $\frac{1}{3.5}=\frac{2}{7} = 0.\dot{2}8571\dot{4}$, or a bit less than $\frac{1}{3}$. We can see in Figure 1 that the probability of landing on the nth square does appear to be settling down towards this value.

For a free lesson plan based on this problem, see Foster (2017).

Reference

Foster, C. 2017 'Hit Ten', *Teach Secondary*, in press. To access this free lesson plan, go to **http://www.teachwire.net/teaching-resources/mathematics.**

Note

1. I originally heard it from Professor Malcolm Swan at the University of Nottingham.

Table 1 All the possible ways of reaching the 10th square and the associated probabilities

No. of throws	Scores obtained	No. of permutations	Total no. of ways	Probability
2	6 4	2	3	$\frac{3}{6^2}$
	5 5	1		
3	6 3 1	6	27	$\frac{27}{6^3}$
	6 2 2	3		
	5 4 1	6		
	5 3 2	6		
	4 4 2	3		
	4 3 3	3		
4	6 2 1 1	12	80	$\frac{80}{6^4}$
	5 3 1 1	12		
	5 2 2 1	12		
	4 4 1 1	6		
	4 3 2 1	24		
	4 2 2 2	4		
	3 3 3 1	4		
	3 3 2 2	6		
5	6 1 1 1 1	5	126	$\frac{126}{6^5}$
	5 2 1 1 1	20		
	4 3 1 1 1	20		
	4 2 2 1 1	30		
	3 3 2 1 1	30		
	3 2 2 2 1	20		
	2 2 2 2 2	1		
6	5 1 1 1 1 1	6	126	$\frac{126}{6^6}$
	4 2 1 1 1 1	30		
	3 3 1 1 1 1	15		
	3 2 2 1 1 1	60		
	2 2 2 2 1 1	15		
7	4 1 1 1 1 1 1	7	84	$\frac{84}{6^7}$
	3 2 1 1 1 1 1	42		
	2 2 2 1 1 1 1	35		
8	3 1 1 1 1 1 1 1	8	36	$\frac{36}{6^8}$
	2 2 1 1 1 1 1 1	28		
9	2 1 1 1 1 1 1 1 1	9	9	$\frac{9}{6^9}$
10	1 1 1 1 1 1 1 1 1 1	1	1	$\frac{1}{6^{10}}$
Total		492	492	$\frac{17\,492\,167}{60\,466\,176}$

Table 2 The numbers of ways of reaching square n in t throws.
(Shaded cells show deviations from the numbers in Pascal's triangle.)

Number of throws (t) \ Number of squares (n)	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	0	0	0	0
2	0	1	2	3	4	5	6	5	4	3
3	0	0	1	3	6	10	15	21	25	27
4	0	0	0	1	4	10	20	35	56	80
5	0	0	0	0	1	5	15	35	70	126
6	0	0	0	0	0	1	6	21	56	126
7	0	0	0	0	0	0	1	7	28	84
8	0	0	0	0	0	0	0	1	8	36
9	0	0	0	0	0	0	0	0	1	9
10	0	0	0	0	0	0	0	0	0	1
Total Prob	$\frac{1}{6}$	$\frac{7}{36}$	$\frac{49}{216}$	$\frac{343}{1296}$	$\frac{2401}{7776}$	$\frac{16\,807}{46\,656}$	$\frac{70\,993}{279\,936}$	$\frac{450\,295}{1\,679\,6161}$	$\frac{2\,825\,473}{10\,077\,696}$	$\frac{17\,492\,167}{60\,466\,176}$

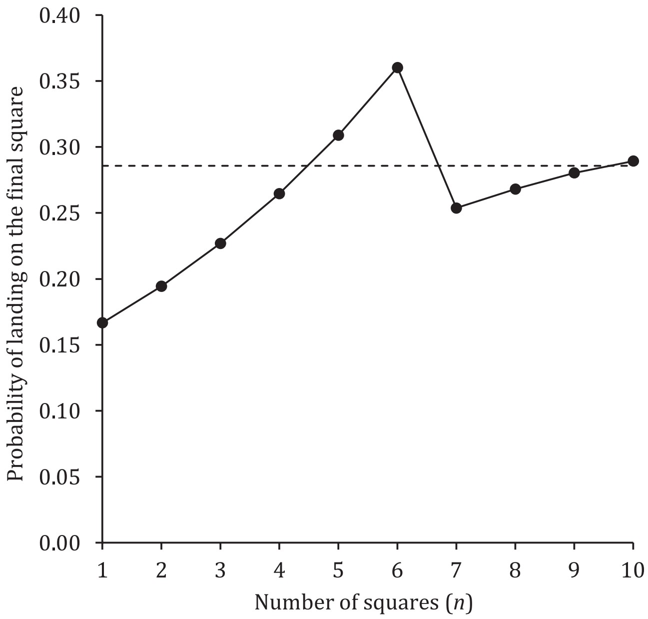

Fig. 1 The probability of winning games with different numbers of squares (The dashed line shows the limiting probability of $\frac{2}{7}$ as the number of squares tends to infinity.)

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