Revisiting 'Four 4s'

by **Colin Foster**

If any mathematics task can claim to be a classic, then this one can:

What numbers can you make from four 4s?

I remember being introduced to it when I was doing my PGCE, and I have used it, and variations on it (e.g., Foster, 2017), ever since, with pupils in school, as well as (more recently) with PGCE students at university.

Presented like this, it would certainly be classed as an 'open-ended task' – i.e., a problem with many possible right answers. I have been thinking about the pros and cons of open-ended tasks like this. One reason this task has multiple possible answers is that it is not "well-posed". Given this task, you might respond with more questions, rather than with an answer: What do you mean 'make'? What are the rules? Do we have to use *all four* of the 4s each time? What exactly are we allowed to *do* with the 4s; what operations are permitted? Can we use square roots (or other roots)? Can we use factorials? Can we *concatenate* and do things like $\frac{44}{4} = 11$? What about $\frac{4}{.4} = 10$ or $\frac{4}{.\dot{4}} = 9$ or $\frac{4}{4\%} = 100$ – are they allowed? You might be reluctant to do anything with this problem until these questions have been answered. You don't want to start playing a game until you know exactly what's allowed, because you don't want rules to be invented or changed while you're part way through!

So, should the teacher carefully clarify all these things at the outset? With a task like this, if the teacher does this, it can feel as though they are killing off some of the nicest aspects of the task. In my few questions above, I have spoiled some nice thinking that you might have otherwise engaged in for yourself. I assumed that, for a teacher audience, most readers would be likely to have encountered this problem before and have played around with it, but with students I would probably not want to raise those questions before they had had a chance to think about it. I would prefer *them* to think and argue about these issues as they occur to them. So there is an opportunity to engage students in problem *construction* as well as problem solving, and this can be one function of an open-ended task: first decide what the task *is*.

For me, this gets to the pedagogical purposes of using a task like this. Do you want to focus students on their ingenuity manipulating numbers within a well-defined set of rules? If so, you might want to nail down the rules quite precisely at the start, and restrict operations to those that you want the students to practise. This maximises scope for creative ingenuity – it is still highly open-ended – but within prescribed limits of allowed operations. Maybe you want to use the task to support some work on priority of operations? In that case, you might want to allow a wider range of different operations – but would including something like concatenation be helpful, or just a distraction? Or maybe you want to see what mathematical operations and processes students can think of, bring together and use creatively? If so, you might want to leave it posed as openly as possible and allow the discussion about what's allowed to emerge naturally. This leads me to the conclusion that 'four 4s' is not 'a task' but a scenario that can form the basis for a whole range of quite different, related tasks with different pedagogical purposes and outcomes. There isn't a best version of 'four 4s' – it depends what you want it to do for your students.

I have often used this task to introduce some work on priority of operations (see Foster, 2008). Students come up with their ideas and write them down, but when another student tries their calculation they get a different answer because they apply the operations in a different order; e.g., someone says that $4 + 4 + 4 \div 4$ is 9, but someone else thinks it's 3. Calculators come out, and may disagree with each other too! Students say, "No, that's not what I meant!", and the ambiguity of notation and its interpretation becomes an obvious communication problem, which creates a need for some agreed conventions.

Whenever I have used four 4s in a very open-ended way, without specifying the rules, at some point a student has asked something like, "Can we use square roots?" and, suddenly, "square roots, square roots, …" reverberates around the room, because square roots are pretty useful in this task, but until that point no one had considered using them. I have also often noticed students becoming struck by apparently basic things like $\frac{4}{4} = 1$, and not 0. Or that when they have made 4 + 4 = 8, but the rule is that they have to use *all four* 4s, they can find ways to use up or 'waste' the other two 4s by doing something like 4 + 4 + 4 – 4 = 8, which can lead to discussion about inverses. Are there any other ways we could 'waste' those two superfluous 4s? Maybe $(4+4)\times\frac{4}{4}$ or $(4+4)^{\frac{4}{4}}$ or even 4 + 4.4 – .4. This is creative thinking that can develop students' fluency with symbols, notation and number in an interesting and purposeful context (Foster, 2014). The task is not really about making these numbers – we don't need the numbers! – it's about students thinking creatively and using their ingenuity.

I have never been quite sure what to make of the 'complete' solutions to this problem. For example, Dirac gave a complete solution to the "four 2s" version, which turns out to be easy to adapt to four 4s (Farmelo, 2009). His method involves logarithms and is based on the idea that *n* nested radical signs followed by a single number 4 (i.e., square rooting 4 a total of *n* times) will produce $4^{\left(\frac{1}{2}\right)^n}$:

$$\underbrace{\sqrt{\sqrt{\sqrt{\sqrt{\ldots\sqrt{4}}}}}}_{\leftarrow n \text{ roots} \rightarrow} = 4^{\left(\frac{1}{2}\right)^n}$$

This turns out to be very useful, because it means that

$\log_4 \sqrt{\sqrt{\ldots\sqrt{4}}} = \left(\frac{1}{2}\right)^n$,

so $\log_{\frac{1}{2}} (\log_4 \sqrt{\sqrt{\ldots\sqrt{4}}}) = n$.

And then, luckily, we can express the '$\frac{1}{2}$' base of this logarithm using the two remaining 4s, as $\frac{\sqrt{4}}{4}$, and we are there! So $\log_{\frac{\sqrt{4}}{4}} (\log_4 \sqrt{\sqrt{\ldots\sqrt{4}}}) = n$, which uses four 4s, and so generates *any* positive integer *n* by simply including that many root symbols.

For example,

$\log_{\frac{\sqrt{4}}{4}} (\log_4 \sqrt{4}) = 1$

$\log_{\frac{\sqrt{4}}{4}} (\log_4 \sqrt{\sqrt{4}}) = 2$

$\log_{\frac{\sqrt{4}}{4}} (\log_4 \sqrt{\sqrt{\sqrt{4}}}) = 3$

etc.

What do you make of this? It's undoubtedly clever, as are related versions such as

$$n = \frac{\log 4}{4^{\frac{1}{\sqrt{4}}} \log 4 \sqrt{\sqrt{\sqrt{\sqrt{\ldots\sqrt{4}}}}}}.$$

These solutions certainly destroy the problem as envisaged above, and appear to be in quite another league. But are they? For me, they raise different questions, like 'How could I adapt these formulae for "five 5s"?' In mathematics, there is always still something to explore, but here it is now quite different from the original scenario. However, I think that the ingenuity involved in creating the general solution is not necessarily greater than that used to produce some of the *ad hoc* solutions for single values of *n*. (*See* **http://paulbourke.net/fun/4444/** *and* **www.dwheeler.com/fourfours/fourfours.pdf** *for very complete sets of solutions, including some very creative ones.*)

Facility with indices and logarithms and nested roots is great. But so is the kind of facility with numbers that notices things like $\frac{4}{.4} = 9$. There is generalisation here too, of course, if students note that, abusing notation slightly, $\frac{n}{.\dot{n}} = 9$, for integer *n* from 1 to 9 (Note 1). So I think some of these sorts of ideas and solutions are just as clever and neat and valuable as any 'complete' solution.

Note

1) As an aside, the fact that $\frac{9}{.\dot{9}} = 9$ completes the pattern suggests that $0.\dot{9} = 1$ might be a good idea!

Reference

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