Starting with Completing the Square

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Within any particular content area, is it better for students to have one general method, that always works, even if sometimes it is a bit inefficient, or to have several different methods at their disposal, and choose each time the most appropriate one for every particular scenario? This is not an easy question to answer. The issue is not just which of these two options might be more likely to lead to the student being able to get to the right answer quickly and efficiently – it may be that the action of choosing from among a range of methods is an important feature of working mathematically, and something we need to help students to do.

Probably, there is no general answer that applies within all content areas and to all students at all times. But there are lots of topics where this choice arises, across all age ranges. As an example, consider solving quadratic equations. This might be done by inspection, factorisation, graphing, completing the square, or use of the quadratic formula. The last three methods can be used in *every* case; the first two methods are suitable only in certain 'nice' cases, but in those cases they can be more efficient.

Suppose that we decide – or it is imposed on us by a specified curriculum that we have to follow – that we want our students to know *all* of these methods. We will want the students to understand the pros and cons of the different methods and how this plays out for different quadratic equations. We will want them to become critical in selecting a suitable method and justifying their choices. But, even if you are teaching all of the methods, you still have to decide on an *order* in which to teach them: most general to least general, or the reverse, or some other rationale? I think often it is assumed that 'easier first' or 'less general first' must be best, but I am not sure that they are always the same as each other, or that they are necessarily helpful principles. Building up to the quadratic formula as the grand finale – what Barry Garelick (2021, p. 68) has called the "death march to the quadratic formula" – risks leaving students wondering why they had to go through all of those other methods, when there was a formula they could have simply substituted into all along (see Foster, 2014).

The move from solving *linear* equations, where we apply identical operations to both sides to isolate the unknown and find a *single* solution, to solving *quadratic* equations, where we use strange things like the 'zero product property' (Note 1), and end up with *two* solutions, is a big step. For many students, the transition does not feel seamless, and quadratic equations appear to exist in a separate box from everything else that they know about equations. How can we build more coherence around 'solving equations', regardless of what kind they are? I think that the common practice (at least in England) of teaching the factorisation method first may not be ideal for this.

When in the curriculum does a student solve their *first* quadratic equation? This is a bit difficult to answer. When beginning to use Pythagoras' Theorem to find the missing side of a right-angled triangle, students will end up with an equation like $c^2 = 25$, and they probably follow this by writing something like '$c = 5$ cm' (Note 2). The negative solution is discarded, because we are finding a length, and lengths are positive, but this is often not particularly highlighted, since the focus is on triangles, not quadratics. So, the fact that we are solving a *quadratic* equation here is not the centre of attention. The same thing applies when students, at perhaps about the same age, are using πr^2 to find the radius of a circle, given its area. This may be another candidate for the first time they are solving a quadratic equation, but again the final square-rooting step gives a single positive answer, because again a radius is a length. The other early occasion when, in a sense, students might be solving a quadratic equation is in number work on factors, squares and square roots, where the fact that the square root symbol $\sqrt{}$ is conventionally taken to mean the *positive* square root means that we often talk as though numbers have a *single* square root, and simply say things like '*the* square root of 25 is 5', without further comment. All of these examples conspire to conceal the fact that quadratic equations (normally) have *two* roots.

So, it is usually much later on when students are asked to find *both* roots of a quadratic equation, and there are many ways to approach this. I like to begin with the equation $x^2 + 2x = 15$ and ask them to try to solve it. I find that students tend to start by making invalid algebraic moves, such as square-rooting the first term, and the 15, but conveniently leaving the $2x$ as it is, or possibly halving it, to obtain something like $x + x = \sqrt{15}$. What they are doing is wrong, but it is 'the right kind of thing', so, rather than telling them that they have 'broken some rules of algebra', I prefer to ask them to check their answers – which they usually realise that they can do by substituting them back into the equation – and they discover that their answers don't satisfy it. This provides a useful moment to clarify what it means to solve an equation. Students usually say, "We have to find out what x is", and this can be tightened up into, "We have to find all the possible values of x that will satisfy the equation". In this way, I distinguish between 'finding solutions' (which could be by trial and improvement, inspection or graphing) and 'solving the equation', which means *finding all of the solutions and knowing that you've got them all*. For linear equations, where there is only one solution, this distinction may seem pedantic, but it seems to me an important one.

This checking to see if their solutions satisfy the equation tends to prod students into proceeding by trying some more numbers, to see if they can find a solution by trial and error. It doesn't take long for them to discover that $3^2 + 2 \times 3 = 15$, and they are usually pleased to have 'solved it'. So, then I ask, "Have you solved it, or have you just found *a* solution?" This is not the first time that they have met problems that have more than one solution (e.g., "What are the factors of 12?"), so the idea that there could be more than one answer is not in itself difficult. Yet, in this case, students are often quite convinced that there can't be any other solutions. As they experiment with other numbers, it seems obvious that, since $2^2 + 2 \times 2 < 15$ and $4^2 + 2 \times 4 > 15$, 3 must be the only value that could work.

At this point, it would be possible to give them a hint, and say something like, "How about trying *negative* numbers?", but for me that is too leading and not a particularly mathematical way of proceeding. Instead, I prefer to ask students whether they are sure that numbers *greater than 3* will *always* give answers *greater than 15* (Note 3). They can usually argue that, as x gets bigger, both x^2 and $2x$ will get bigger, so "for all $x > 3$, $x^2 + 2x > 15$". I would be happy with this just stated in words, not symbols, and I would try to get students to explain this in their own words to each other, so that everyone engages with articulating this reasoning, not just listening to it.

Now, if $x < 3$, will $x^2 + 2x$ *always* be *less than 15*? This is more complicated. This feels like a more natural way for students to explore, and, as a result of this prompt, they always end up trying negative numbers and realising, with surprise, that "It goes back up again!" If $x = -100$, say, then $x^2 = 10{,}000$, not $-10{,}000$, and that is much larger than the magnitude of the $2x$ term is going to be, even though that term is negative. And so the total is clearly going to be *large and positive*. It takes time for students to be comfortable articulating this kind of 'qualitative' argument, but I think it is highly mathematical to spend time on this kind of thing, so I would not be tempted to rush this.

Of course, it would have been possible to have introduced the *graph* of $y = x^2 + 2x$ sooner, possibly from the beginning. But, before sketching the graph, I would like students to be able to argue that if $2^2 + 2 \times 2 < 15$ and $(-100)^2 + 2 \times (-100) > 15$ then somewhere between 2 and -100 there must be a value of x (i.e., a second one) for which the left-hand side is *equal to 15*. This is an informal, intuitive application of the *intermediate value theorem*, and strictly this makes assumptions about the continuous nature of the function and the completeness of the reals. This is something I might or might not point out later on, but I would be fine with students not thinking about it at this point. I have never yet had a student suggest here that the values could *jump over* 15 and miss it out, but if they did I would be delighted to have everyone give it some thought.

By this point, the students have a solid reason for searching negative numbers for another solution, and they find $x = -5$ before too long. Then, it is really nice to sketch the graphs of $y = x^2 + 2x$ and $y = 15$, and revisit everything we've done to see the $x = 3$ and $x = -5$ solutions, and how the graph 'goes back up again' halfway between them (i.e., at $x = -1$). But, I would delay introducing the graph until now, as once the graph is on the board a lot of the thinking is redundant, and I would prefer the students to do more reasoning before revealing the graph.

This takes a whole lesson, and is how I would start with introducing quadratic equations. I think it is much more common to get into quadratic equations by teaching the factorisation method, using the zero-product property (Note 4). That gives you a much quicker payoff, in terms of actually being able to *solve* certain equations. But, for me, the factorisation method naturally belongs much later. The key things I want students to understand initially are what it means to *solve* an equation (as opposed to *finding solutions* in a hit-and-miss fashion) and how we can reason our way to a convincing conclusion that we've found *all* of the solutions. I want them to see that not all algebraic graphs are straight lines, and some of them cross the x-axis more than once. I might remark that these equations are called 'quadratic', because of the x^2 term that they contain, but I wouldn't make a big deal out of this yet.

When people describe what they consider to be a 'nice' lesson, they often leave it there, and I find myself wondering "How would you follow-up on that? What would come next?" It's perhaps not too hard to invent nice, standalone lessons, but the real challenge is to integrate them into a connected experience for students (see Foster, 2020). So, I'll sketch out now how the trajectory might unfold over the rest of this unit.

In the next lesson, I would go back to the same equation, $x^2 + 2x = 15$, and look at how we can solve it *algebraically*, but I wouldn't do this by factorising, as, at this stage, to me that feels like 'a trick'. Subtracting 15 from both sides is a very odd thing to start by doing. This has little overlap with anything students have previously learned about solving equations, where the aim is to isolate x, not 'put everything on the left-hand side'. So, telling students that we are going to start by subtracting 15 from both sides would have to be a 'Trust me, I'm a teacher' kind of moment, and those are things that I prefer to avoid whenever possible. Instead, I would start by teaching the *completing the square method*.

In Japan, completing the square is often the first method taught for solving quadratic equations, and it has the advantage that it is kind of the most logical thing that you would do, if you were building on your thinking from solving *linear* equations, so there is a coherence to beginning here. It also has the advantage that, unlike factorisation, it works for *all* quadratic equations, and it doesn't involve 'guessing some numbers that will work', as happens in the factorisation method, which can feel to students more like finding solutions by trial and improvement than truly *solving* an equation.

Completing the square has a reputation for being a difficult, scary method that sometimes only gets taught to the highest-attaining students (e.g., the top 40% or so in England who do higher-tier GCSE), and that even they may quickly abandon in favour of 'the formula'. However, I don't think this has to be the case, if we take our time and introduce it early. And, during all the time spent working on completing the square, students are practising things like expanding brackets that they need to practise anyway. If you introduce completing the square after students already have other methods, then they are likely to think, "This seems harder than factorising!", and they may say that they prefer the other methods, not realising that those other methods will not always work for every quadratic equation.

If students have had plenty of prior experience expanding brackets of the form $(x + \alpha)^2$, they should spot the left-hand side of $x^2 + 2x = 15$ to be 'nearly' $(x + 1)^2 \equiv x^2 + 2x + 1$. It is only a constant addition away from being $(x + 1)^2$, so I find that students are very happy to write

$$(x+1)^2 - 1 = 15$$
$$(x+1)^2 = 16$$
$$x + 1 = \pm 4.$$

So, either $x = 3$ or $x = -5$.

The big idea is to make a perfect square ("something, squared, equals something"), as on line 2, which is then perfectly set up to square-root both sides.

The first line is a good opportunity to ask why the $+1$ and -1 cannot be cancelled out. A common mistake on the second line is to write 14, rather than 16, on the right-hand side, but these are the same sorts of issues that arise with solving linear equations (compare with solving something like $\frac{x+1}{2} - 1 = 15$). The only *differently-tricky* part really is the first line – finding the *square* which is just a constant addition away from the original left-hand side.

My next example would be $x^2 + 2x = 16$. I would ask the students to try to find the solutions by trial and improvement, and they are usually quite good at getting rough answers, or at least locating them between pairs of consecutive integers, now that they know that they should expect two solutions. Then we do it algebraically, as before:

$$(x+1)^2 - 1 = 16$$
$$(x+1)^2 = 17$$
$$x + 1 = \pm\sqrt{17}.$$

So, either $x = -1 + \sqrt{17}$ or $x = -1 - \sqrt{17}$, and we can work out decimal approximations to these on a calculator, and see that they are close to the rough values found by trial and improvement. With a calculator, this is really no harder than solving $x^2 + 2x = 15$.

I would now do a lot of practice on this, with *monic* left-hand sides (i.e., the coefficient of x^2 is 1) in which the coefficient of x is *even,* before making things any more complicated. There is plenty to think about with this for now. Eventually, the extra complexity of relaxing those constraints just adds a little greater fiddliness, and, by the time you have done all of this, students can solve *any* quadratic equation. So, if getting there takes a while, that is OK.

I would go on to teach factorisation as a special case (Note 4), and the quadratic formula as an afterthought, and I would put less emphasis on these. Completing the square is useful in many circumstances where you are not even *solving* a quadratic equation, such as for finding the minimum point of the graph $y = x^2 + 6x + 1$ (without calculus), by writing $y = (x + 3)^2 - 8$ and concluding that the minimum is at $(-3, -8)$. Factorising and using the zero-product property is an important idea, but I would bring it in later. And the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

is not the panacea that students often think it is. I find that they often say it is their favourite method, but frequently they end up with the wrong answer, perhaps because they misapply the order of operations or miscalculate b^2 when $b < 0$. More importantly, I am reluctant to reduce mathematics to plugging numbers into a given formula (see Foster, 2014). The main purpose of teaching quadratic equations isn't really to be able to get the answers. If that were the main purpose of teaching equations, then we would presumably teach a formula for *linear* equations too; i.e., the solution to

$$ax + b = cx + d \text{ is } x = \frac{d - b}{a - c}, \text{ when } a \neq c.$$

But, instead, we teach methods like balancing, because they are important to an understanding of what an equation *is* and what it means to solve it, and support processes like rearranging equations, where there is no numerical answer to find. Similarly, completing the square gives important insight into quadratic equations that merely substituting into 'the formula' does not. I want to avoid a view of learning mathematics that reduces everything to 'Find the right formula on the formula sheet and then substitute in'.

So, I think completing the square is a powerful and important method that deserves a higher status within the teaching of quadratic equations. It offers a better connection with what has gone before and a stronger basis for what is to come. So, why not begin with it?

Notes

1. The *zero-product property* is the fact that if $ab = 0$ then either $a = 0$ or $b = 0$.
2. I will avoid here the issue of whether, in mathematics, c should be a dimensionless number – i.e., 'the length in centimetres' – or, as in science, 'the length', including the units.
3. Here, I am subtly shifting to treating the left-hand side as a function, $f(x) = x^2 + 2x$, but I probably wouldn't draw attention to this explicitly.
4. I mean: $x^2 + 2x - 15 = 0$, which leads to $(x + 5)(x - 3) = 0$, so $x = -5$ or 3.
5. One way might be to factorise the left-hand side of $x^2 + 2x = 15$, to give $x(x + 2) = 15$. If we suppose that x is an integer, then $x + 2$ will be also. So, then we are seeking two integers, 2 apart, with a product of 15, and students will quickly suggest 3 and 5, meaning that $x = 3$ works, since $3 \times 5 = 15$. But it is harder to spot that $x = -5$ also works for this, since $(-5) \times (-3) = 15$, and -3 is 2 more than -5. (Once students have seen this, they may be able to solve integer equations like this, without having to transform them into a zero right-hand side.) Then, changing the equation into $x^2 + 2x - 15 = 0$ can be seen as a reasonable extension of this process. The other approach to quadratic factorisation that I like is to begin as for completing the square, by going from $(x + 1)^2 - 1 = 15$ to $(x + 1)^2 - 16 = 0$ and then to $(x + 1)^2 - 4^2 = 0$. Then, you use the difference of two squares to rewrite this as $(x + 1 + 4)(x + 1 - 4) = (x + 5)(x - 3) = 0$. Here, there is no 'guessing what the factors might be', and factorisation is just presented as a variation on completing the square. In her plenary at the 2022 Easter Conference, Jo Morgan suggested that this approach to factorisation, which may seem strange to us, may be common in other countries.

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