Terminal Digits and Prime Numbers

By **Colin Foster**

In a secondary school mathematics lesson, students were making and evaluating conjectures to do with the natural numbers and multiples. For example:

If a number ends in a 5, it is a multiple of 5;
if a number ends in a 2, it is a multiple of 2.

In both cases, students had noticed (without using this language) that these were sufficient but not necessary conditions. This led them to question for which values of n this general statement is true:

If a number ends in a n, it is a multiple of n.

They agreed that it was true for $n = 1$, and argued about whether

If a number ends in a 0, it is a multiple of 10

was or was not a valid member of this family.

Someone suggested:

If a number ends in a 3, it is a multiple of 3.

This was quickly disproved with counter-examples (13 and 23). A student commented: *That's because they're prime, and prime numbers aren't divisible by anything.* This led to the revised conjecture:

If a number ends in a 3, it is either a multiple of 3 or prime.

This seemed much more plausible, and students set about testing it on a few examples:

Number	Multiple of 3?	Prime?
3	✓	✓
13		✓
23		✓
33	✓	
43		✓
53		✓
63	✓	
73		✓
83		✓
93	✓	

There was some discussion about the number 3 itself. Does 3 'end in a 3' if it also 'starts in a 3' and is 'just a 3'? Is 3 itself a 'multiple' of 3, if the multiplier is just 1, and so it appears 'not to have been multiplied'? And does the word 'or' in the conjecture include the possibility that *both* are satisfied?

I do not think I had previously considered this conjecture, but it was obvious to me that it must be false, because divisibility by 3 is easy to establish using the digit sum property (the digital root of a number is a multiple of 3 if and only if the number is a multiple of 3). This meant that if this conjecture were true then we could trivially generate arbitrarily large prime numbers. For example, we could write down a number such as 1,000,000,003 and immediately declare it prime. Indeed, $10^n + 3$ would be prime for all natural n. Since finding large prime numbers is known to be extremely difficult, it was clear to me that this conjecture could not possibly be true. But this didn't seem like a very convincing argument (see Note).

The lesson left me reflecting on the question of how far students should continue searching for a counterexample before supposing that the statement is probably true and trying to prove it. In fact, the smallest counterexample for this conjecture is 133, because 133 is 7×19, but no one carried on through examples that far. Should they have? Perhaps a spreadsheet and/or a table of prime numbers would have been useful. This would have revealed that counterexamples to this conjecture become much more plentiful as the numbers become higher.

This episode also got me thinking about different kinds of mathematical knowledge that can be sort of useful to know, but which would not be 'serious' enough to be explicitly taught (Foster, 2011). For example, following this lesson, I found this false conjecture quite memorable, along with its smallest counterexample of 133, and this was subsequently slightly useful in remembering which numbers are prime. Effectively, I modified the students' conjecture into the true statement:

If a number <u>less than 133</u> ends in a 3, it is either a multiple of 3 or prime.

Before this, I would often struggle to remember whether numbers such as 83 or 103 were prime or not, and would have to mentally check. But now, with this (silly) theorem, I can ascertain this simply by seeing at a glance (by digit sum) that 83 and 103 are not multiples of 3, and therefore (since they are less than 133) they must be prime.

This reminded me of the story of the so-called 'Grothendieck prime' (Jackson, 2004), in which the mathematician Alexander Grothendieck is reported to have mistakenly offered 57 as an example of a prime number. This led me to consider whether a conjecture such as:

> *If a number less than M ends in a 7, it is either a multiple of 7 or prime*

might be useful in practice, in a similar way to the one about terminal-3 numbers, for separating the terminal-7 primes from the terminal-7 non-primes. However, it does not seem to be, because (i) divisibility tests for 7 are more complicated than those for 3, and (ii) $M = 27$, which is far too low to be practically useful. Similarly, for numbers ending in a 9, the equivalent M is 39, which is again far too low to be of use.

Note

In fact, $1{,}000{,}000{,}003 = 23 \times 307 \times 141{,}623$, and so is not prime.

References

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