The Directionality of the Equals Sign

By **Colin Foster**

It is often observed that students prefer to read or write an equation in the form 2 + 5 = 7, with the operation on the left side and the 'result' on the right, rather than in the form 7 = 2 + 5, where the operation takes place on the right side and the 'result' is on the left. And they are happier being asked to complete a problem like 2 + 5 = ☐ than a problem like ☐ = 2 + 5 (for a discussion, see Jones & Pratt, 2012). This is often taken as evidence of a misconception concerning the equals sign. Equals is an equivalence relation, and one of the properties of equivalence relations is that they are *symmetric*, meaning that the statement $A = B$ is exactly equivalent to the statement $B = A$. The students' preference for one form over the other shows that they fail to appreciate this.

I am not sure that this is quite the right way to think about this, and I think it could be an unfair misinterpretation of what is going on. The symmetric property of equivalence relations is that *if $A = B$ then it is also true that $B = A$*, and vice versa. The two statements imply each other, so we could write:

$$A = B \Leftrightarrow B = A.$$

So, I agree that if a student thinks, for example, that 2 + 5 = 7 is true, but 7 = 2 + 5 is false, then they have a problem with their understanding of this property. But that is not the same thing as saying that students should never, in any situation, have a preference for one of these statements over the other, or that it doesn't matter which way round we write equations. There is often a definite left to right directionality to equations.

For example, if someone asked me to "Let $x = 7$" in the equation $y = x + 7$, I would write $y = 14$, whereas if they asked me to "Let $7 = x$", I might be a bit confused whether I should instead write $y = 2x$. Of course, since $14 = 2x$, the value of y would be the same either way, but what I write down would be very different. A more extreme example of this would be substituting $x = x^2 - 1$ in the equation $y = x^2 - 1$, where I would write $y = (x^2 - 1)^2 - 1$, whereas if I were substituting $x^2 - 1 = x$ (the same equation, written the other way round) into $y = x^2 - 1$, I would probably write $y = x$. Again, of course, it is true that $(x^2 - 1)^2 - 1 = x$, but we have ended up making a quite different-looking substitution in each case, and in this second case we have gained two new solutions ($x = 0$ and $x = -1$), which we didn't have before, and which don't satisfy one of our original equations, so our algebra has taken a very different path.

In secondary mathematics, this kind of discussion often comes up in the context of solving equations. If a student writes

$$10 = x + 3$$
$$7 = x$$

and then, before finishing, reverses the final line to give $x = 7$, they are accused of not properly understanding the meaning of the equals sign. Don't they realise that $7 = x$ and $x = 7$ are making exactly the same statement? Here, the student feels that $7 = x$ is saying something about 7, whereas $x = 7$ is saying something about x, so they prefer to write the latter, since they were asked to 'Solve for x', and they understand that to mean that x must be isolated *and on the left side* (see Note 1).

But the teacher thinks this is just a misunderstanding, and they might try to address this by saying something like "If 7 equals x then you know that x must be equal to 7, so you've finished at line 2; line 3 is redundant!" But, if all the teacher is really claiming here is that line 3 follows inexorably from line 2, then that is equally true of line 2. It follows inexorably from line 1, where $10 = x + 3$. Objecting to writing down statements that are 'merely' equivalent is kind of objecting to the whole rationale of solving equations. Line 3 and line 2 are both mathematically equivalent to line 1, and to infinitely many other such statements, such as $14 + x = 3x$. By 'solving the equation', we mean stating the value(s) of x *explicitly*, and I think it is reasonable for the student to feel that $x = 7$ does this better than $7 = x$. This kind of issue is perhaps particularly apparent in a topic like 'rearranging equations'. I think teachers might disagree about whether, when given the instruction 'Make x the subject of $y = x + 3$', it is sufficient to write $y - 3 = x$, or it would be necessary to write $x = y - 3$, so that the x is on the left side. After all, wouldn't we say that y is the subject in $y = x$, but x the subject in $x = y$ (Note 2)?

In mathematics, we often indicate something important by the order in which we write the two sides of an equation (Note 3), and this is particularly so when there is more than one equals sign on the same line. For example, writing

$$2 \times 5 + 5 = 3 \times 5 = 15$$

suggests collecting together 2 lots of 5 and 1 lot of 5 to make 3 lots of 5, whereas switching the order to

$$3 \times 5 = 2 \times 5 + 5 = 15$$

suggests the opposite: splitting the three 5s into a pair and a single 5, making a 10 and a 5. As another example, if you wished to draw attention to the priority of operations, you might write something like this:

$$3 \times 4 + 5 = 12 + 5 = 17$$
$$\text{whereas } 3 \times (4 + 5) = 3 \times 9 = 27 \text{ and } 17 \neq 27.$$

However, if you wanted to, you could abbreviate this to one line:

$$27 = 3 \times 9 = 3 \times (4 + 5) \neq 3 \times 4 + 5 = 12 + 5 = 17$$

and, here, again, reordering the equals signs would destroy the coherence of the argument. This is generally the case with one-line proofs – they are intended to be read left to right, and everyone understands that.

So, I think it is not ridiculous for a student to think that $3 \times 4 = 12$ is telling us something about 3 and 4, whereas $12 = 3 \times 4$ is telling us something about 12. They are both true, but they are not 'the same thing'. We would probably be more likely to write $12 = 3 \times 4$ in the context of factorising, followed by things such as $12 = 2 \times 6$, whereas we would be more likely to write $3 \times 4 = 12$ in the context of multiplying, followed by $3 \times 5 = 15$ or $30 \times 40 = 1200$. I think in many parts of school mathematics the equals sign is treated as having a left to right directionality, and so, alongside understanding the symmetry of the equals sign, students also need to recognise this.

Notes

1. Not to mention the fact that some online homework software will mark things like '$7 = x$' wrong unless it is turned around to '$x = 7$'.
2. One reason for seeing 'switching the sides' as a bad habit is concern about students doing this when solving inequalities, like $10 > x + 3$, where $7 > x$ gets incorrectly turned around into $x > 7$ instead of $x < 7$.
3. An obvious example of this would be the statement that I made at the start, that $A = B \Leftrightarrow B = A$. Changing the sides for the second equation would make this into $A = B \Leftrightarrow A = B$, which would be trivial. Ironically, the order of the sides matters if we want to make a statement expressing that the order of the sides doesn't matter!

Reference

Jones, I., & Pratt, D. 2012 'A substituting meaning for the equals sign in arithmetic notating tasks', *Journal for Research in Mathematics Education* 43 (1), 2-33.

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