THE FLOOR AND CEILING FUNCTIONS

By **Colin Foster**

Two of the most overlooked 'easy' functions in school mathematics are the *floor* and *ceiling* functions. Bringing these functions into the topic of 'rounding', along with their friend, the 'nearest-integer' function, provides lots of opportunities for students to practise rounding, while deepening their understanding of functions and graphs. It also gives students ways to 'be mathematical' in posing and answering their own questions. There is very little that they need to be 'taught' in order to spend quite a lot of time in deep mathematical thought.

The *floor function* of x, written as $\lfloor x \rfloor$, is the *greatest integer less than or equal to* x. For example, $\lfloor 32.4 \rfloor = 32$, $\lfloor 65.7 \rfloor = 65$, $\lfloor -12.6 \rfloor = -13$ and $\lfloor 5 \rfloor = 5$. Everyone knows about this function because of age. Your age is the time you have been alive, floored (or truncated):

$$\text{age} = \lfloor \text{time alive in years} \rfloor$$

Every child knows that, even if it's just one day to your 14th birthday, meaning that you've been alive for 13.997… years, you are still 13 until the day you turn 14. Age is floored, rather than rounded, and flooring is kind of what makes birthdays special. Awareness of this can lead to puzzles that are more interesting (and much harder) than they might at first appear. For example (see Foster, 2016):

Abdul is 10 and Bella is 12.
How old will Bella be when Abdul is 12?

Often, puzzles like this, involving age, require a very strong implicit assumption that, in this case, Abdul and Bella share a birthday, which seems a very unlikely thing to assume without being told. Without making this assumption, the answer here is not simply 14 (Foster, 2016).

The partner of the floor function is the *ceiling function* of x, written as $\lceil x \rceil$, which is the *least integer greater than or equal to x.* For example,

$$\begin{aligned} \lceil 32.4 \rceil &= 33, \\ \lceil 65.7 \rceil &= 66, \\ \lceil -12.6 \rceil &= -12, \\ \lceil 5 \rceil &= 5. \end{aligned}$$

Once you've defined these two functions, then that's all that students need to be told. Everything else, they should be able to figure out – although it won't be easy! They will already be familiar with the *nearest-integer function*, which we can write as $[x]$ and think of as 'rounding to the nearest integer' (Note 1). For example,

$$\begin{aligned} [32.4] &= 32, \\ [65.7] &= 66, \\ [-12.6] &= -13, \\ [5] &= 5. \end{aligned}$$

Looking at how these three functions operate, and what happens when you combine them, can be an interesting and challenging piece of work.

Here are some possible tasks.

1. Always, sometimes or never true?

$$\lceil x \rceil - \lfloor x \rfloor = 1$$

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor \qquad \lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$$

$$\lfloor x \rfloor < \lceil x \rceil \qquad -\lfloor x \rfloor = \lceil -x \rceil \qquad \lceil x \rceil + \lceil -x \rceil = 1$$

$$\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor \qquad \left\lfloor x + \frac{1}{2} \right\rfloor = [x] \qquad \left\lceil \frac{\lfloor 2x \rfloor}{2} \right\rceil = [x]$$

Try switching around floors, ceilings and nearest-integers. What happens?

What other conjectures like these ones can you make and test?

2. Graph sketching

Sketch each of the graphs below for the three cases $f(x) = \lfloor x \rfloor$, $f(x) = \lceil x \rceil$ and $f(x) = [x]$. Resist using graph-sketching software too soon – try to reason first how the graphs will look, and make sure that you have strong conjectures before you test them out using software (Note 2).

$y = f(x)$	$y = f(2x)$	$y = 2f(x)$	$y = f(x) + \frac{1}{2}$	$y = f\left(x + \frac{1}{2}\right)$
$y = f\left(\frac{x}{2}\right)$	$y = \frac{1}{2}f(x)$	$y = \frac{1}{2}f(2x)$	$y = 2f\left(\frac{x}{2}\right)$	$y = 10f\left(\frac{x}{10}\right)$
$y = xf(x)$	$y = f(x^2)$	$y = (f(x))^2$	$y = \frac{f(x^2)}{x}$	$y = f\left(\frac{1}{x}\right)$

What other possibilities can you think of?

For some reason, I always find working with the floor and ceiling functions much harder than I expect to. I make mistakes that seem silly with hindsight. The definitions seem so straightforward and familiar, but, as soon as I start applying them, I find it very easy to get in a muddle. Situations like this – little knowledge needed, but lots of scope to explore – seem valuable for helping students develop as mathematicians. They are definitely 'low-*floor*-high-*ceiling*' tasks (see Kiddle, 2020)!

Notes

1. Students may find it odd to think of these as 'functions'. Their idea of function might be more continuous, and involve a 'proper formula'. But, the essential requirement for a function (a unique, well-defined answer for every input) is satisfied.

2. The relevant functions in *Geogebra* are floor(x), ceil(x) and round(x).

References

Foster, C. 2016 Abdul and Bella. *Symmetry Plus,* 60, p. 4. www.foster77.co.uk/Foster,%20Symmetry%20Plus,%20Abdul%20and%20Bella.pdf

Kiddle, A. 2020 An introduction to low threshold high ceiling tasks. *Mathematics in School,* 49(2), 2–3.

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