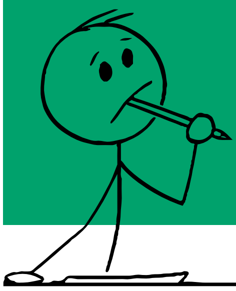
THINKING HARD ABOUT EASY CONTENT: ODD AND EVEN FUNCTIONS



By **Colin Foster**

When asked to define an even function, a student wrote: "A function f for which $f(-x) = f(x)$."

Is this OK? It's quite a good answer – the idea is clearly right. But I would have liked the student to have written something like "for all values of x", or "for all values of x in the domain of f". Am I being too fussy? If the student had used the *identity symbol* (see Foster, 2021) to express this as $f(-x) \equiv f(x)$, then I would have been satisfied. But, as it stands, $f(-x) = f(x)$ could express something that is true for only certain values of x. How might I address this without seeming to be annoyingly pernickety?

Perhaps we need tasks like these:

1. *Can you find an example of a function f such that $f(-x) = f(x)$ for **no** values of x?*
2. *Can you find an example of a function g such that $g(-x) = g(x)$ for **exactly one** value of x?*
3. *Can you find an example of a function h such that $h(-x) = h(x)$ for **exactly two** values of x?*

and so on.

Try to find both simple and more complicated/surprising examples for each of f, g and h.

Can you find methods for generating examples of such functions?

You might prefer to try these tasks yourself before reading on.

The answer to #1 might seem to be 'no' (i.e., there are no such functions), because $0 = -0$, so $f(-0) = f(0)$ necessarily. But, of course, this is true only for functions which contain 0 in the domain. A function that does *not* contain 0 in its domain, either 'naturally' (e.g., $f(x) = \frac{1}{x}$, $x \neq 0$) or 'artificially' (e.g., $f(x) = x$, $x \neq 0$), can provide a suitable example of a function for which $f(-x)$ is *never* equal to $f(x)$. For $f(x) = \frac{1}{x}$, we can see that for all $x < 0$, $f(x) < 0$, and, for all $x > 0$, $f(x) > 0$, so there is no possibility of these ever being equal for any non-zero value of x (Note 1).

Thinking in this way, for #2, it might at first seem that *any* odd function will do, with $g(-x) = g(x)$ for $x = 0$ only. However, even when $x \neq 0$, and so $x \neq -x$, it is still possible for $g(-x) = g(x)$ in addition to $g(-x) = -g(x)$. Indeed, this will happen whenever $g(x) = 0$. So, odd functions with *only one zero* (at the origin) are examples of #2, but other odd functions, with more than one zero, such as $g(x) = x^3 - x = x(x-1)(x+1)$, are *not*. In this case, $g(-1) = g(1) = 0$, in addition to $g(-0) = g(0) = 0$. So, for this function, there are *three* values of x for which $g(-x) = g(x)$, and these are precisely the three zeroes of the function {-1, 0, 1}. The function $g(x) = \sin x$ is a more extreme example of this, with *infinitely many* zeroes, and so infinitely many values of x which satisfy $g(-x) = g(x)$. This means that $\sin x$ is an example of an *odd* function, which, by definition, satisfies $\sin(-x) = -\sin x$ *everywhere*, but which *additionally* satisfies $\sin(-x) = \sin x$ (the student's version of an 'even function' definition) at infinitely-many places (i.e., at all of its zeroes).

What about other non-even (but also not odd) functions (Note 2)? Might they satisfy #2? How might you construct such functions that do? If we try for a cubic, we can write

$$f(x) = ax^3 + bx^2 + cx + d,$$

where a, b, c and d are constants to be determined, and set

$$\begin{aligned} f(x) &= ax^3 + bx^2 + cx + d \\ &= a(-x)^3 + b(-x)^2 + c(-x) + d \\ &= f(-x), \end{aligned}$$

giving $x(ax^2 + c) = 0$. So, a convenient example function ($a = 1, b = 1, c = -1, d = 0$) would be

$$f(x) = x^3 + x^2 - x,$$

which has $f(1) = f(-1) = 1$ (Figure 1). We might wonder whether, for this function,

$$f(x) = x^3 + x^2 - x,$$

$f(-x) = f(x)$ for some *other* value(s) of x (besides $x = -1,\ 0,\ 1$). The answer is no, because $x^3 + x^2 - x = (-x)^3 + (-x)^2 - (-x)$ reduces to $x(x^2 - 1) = x(x-1)(x+1) = 0$, which has only three solutions: $x = -1, 0, 1$.

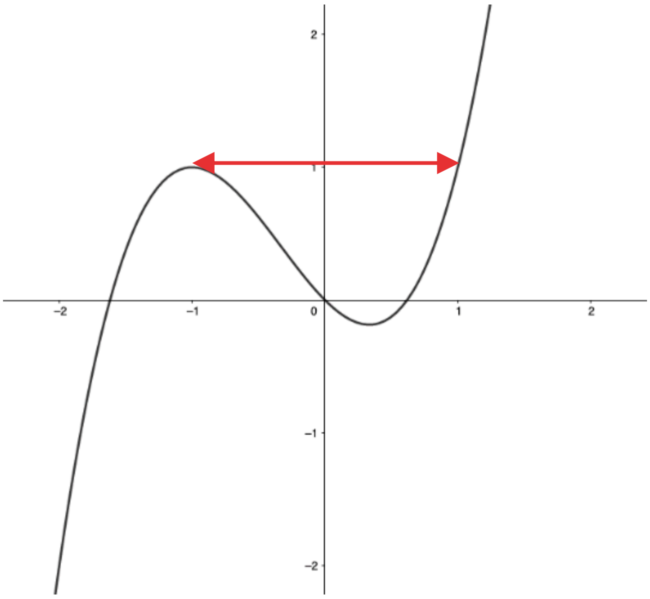


Figure 1. $y = x^3 + x^2 - x$, with $f(1) = f(-1)$

There is no profound mathematics here – this is just playing around with straightforward definitions and testing their boundaries. But I think that tasks like this could help students to understand more clearly what definitions like those of odd and even functions really mean, and provide valuable opportunities to think both graphically and algebraically.

Notes

1. Do you consider $f(x) = \frac{1}{x}, x \neq 0$ to be an *odd function*? Does not being defined at $x = 0$ get in the way of this or not?
2. Of course, most functions, like most real numbers, are neither odd nor even. A function not being even *doesn't* imply that it is odd.

Reference

Foster, C. 2021 'Identity crisis', *Scottish Mathematical Council Journal* **51**, pp. 36–37.

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ANOTHER JIGSAW PUZZLE

by **Jenny Sharp**

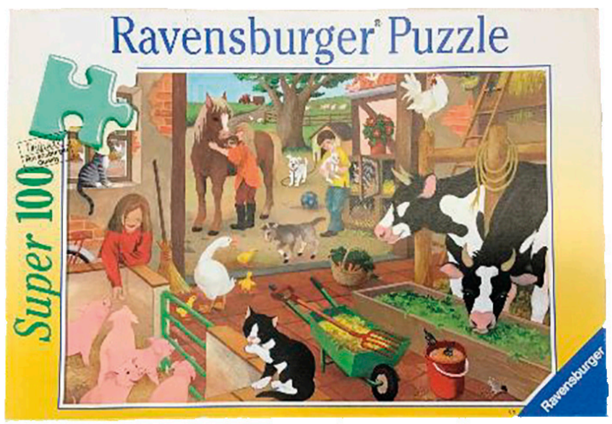
Following on from Paul Stephenson's Jigsaw Puzzle in the March edition of MiS, here is another. How many pieces are there in this rectangular jigsaw puzzle?

If you think 100 – think again. Why *can't* there be 100 pieces? Knowing that there *can't* be 100 pieces might not help in finding the correct answer. However, in problem solving, what you know *cannot* be true is as important as what is true.

I use this as part of a mathematical thinking workshop with Year 5 students. During the day, the students have to come up with helpful expressions – this puzzle led to "It is better to know you're wrong than to think you're right".

Follow up questions:

- What rectangular arrays give the number of pieces to be *about* 100?
- Does that mean 1000 piece puzzles in the same length/height ratio do not have 1000 pieces? What about 2000 pieces? ...



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