Questions Pupils Ask!

Where does the third come from?

by **Colin Foster**

When teaching mathematics, Hewitt (1999; 2001a; 2001b) advocates telling pupils things that are *arbitrary* (e.g. that there are 360° in a full turn) but cautions *against* telling them things that are mathematically *necessary* (e.g. that the sum of the interior angles of a triangle is 180°). The teacher should instead help pupils to work out *necessary* truths for themselves. Simply telling them these things as facts, and expecting them to take them on trust, risks confusing pupils about the nature of mathematical knowledge. One place where it is potentially tricky to follow Hewitt's recommendation is in relation to the formulae for volume and surface area of 3D solids, which pupils need to be able to work with at GCSE before they have had any experience of integration, which provides an elegant way to deduce them. What do we tell pupils at this stage about where these formulae come from? When pupils encounter formulae such as $\frac{1}{3}\pi r^2 h$ for the volume of a cone, how might we deal with questions like "Where does the third come from?"? (Note 1).

When a pupil asked this question in my class, another pupil responded by making a link to the formula for the area of a triangle:

area of a triangle $= \frac{1}{2} \times$ base $\times$ height,

volume of a pyramid $= \frac{1}{3} \times$ base area $\times$ height.

He said that in two dimensions, for area, the fraction is $\frac{1}{2}$, so in three dimensions, for volume, the fraction "must be" $\frac{1}{3}$. I was interested that this response seemed to satisfy the pupil who asked the question, who found this answer very logical. But when I asked, "Why should the number of dimensions be the denominator in the fraction?" no one could answer that. Although calculus would give some insight here, for the pupils it seemed to be an 'explanation' at the level of pattern spotting – perhaps a useful way to remember the fraction, but for me not really an explanation. This reminded me of another occasion, in a different class, when a pupil asked why the surface area of a sphere is $4\pi r^2$ rather than $3\pi r^2$, which seemed logical to him, as when you look at a sphere you see a circle of area πr^2, and you can look at it from three different 'dimensions', making a total surface area of $3\pi r^2$. He wanted to know where the fourth πr^2 came from (Note 2). Both of these connections with 'dimensions' suggest that pupils are eager to make sense of what they are told and want to understand why.

It can often be difficult to find age-appropriate explanations for mathematical results. Is it sometimes reasonable to ask pupils to wait until they have studied more mathematics before giving an answer? That can feel like a cop-out, especially as these formulae were known many centuries before calculus as we know it was formally invented. However, it could be argued that having some unresolved issues is motivating for learning more mathematics. Why bother learning more and more mathematics unless it enables you to make sense of problems that you couldn't (or were harder) before? But not all pupils will go on to study calculus, and I want the subject to make sense for pupils, in some way, at every stage of their learning. So I am motivated to try to find ways of understanding these formulae, even if they fall short of rigorous proofs. I like to believe that there is always a good way to deal with any enquiry based on what the pupil currently knows, but sometimes this can be challenging!

The first thing to notice could be that $\pi r^2 h$ (without the $\frac{1}{3}$) would be the volume of a cylinder of radius r and height h, and that a cone with the same radius and height clearly takes up less space than the corresponding cylinder, so multiplying by a fraction less than 1 seems like a sensible thing to be doing (Fig. 1). Can you see by looking at the cone beside (or inside) the cylinder that the cone is less than half the volume of the cylinder? Perhaps – but certainly not that it is precisely one third of it.

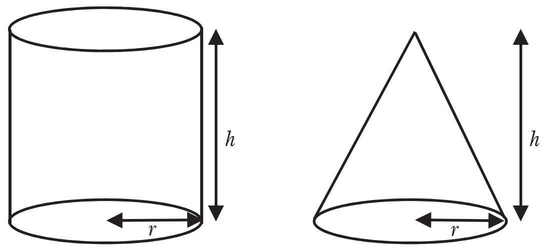

Fig. 1

One approach is to begin with pyramids rather than cones. It is plausible that the volume of a pyramid will be proportional to its base area and to its height. Shearing a pyramid – moving its apex parallel to the base – doesn't change its volume. So to justify that doubling the base area will double the volume, you can imagine two identical square-based pyramids side by side and then shear them so that they fuse together to make one larger pyramid with the same height but twice the base area and twice the volume (Fig. 2). Each horizontal slice of the new pyramid has an area that is the sum of the areas of the slices at the same height in the initial pyramids. So the volume must have doubled.

Likewise, slicing the two starting pyramids into lots of horizontal pieces, and taking slices alternately from each pyramid, would result in a pyramid with the *same* base area but *twice the height* – and clearly twice the volume (Fig. 3). So it is plausible that

volume of pyramid $\propto$ base area $\times$ height,

but what is the constant of proportionality?

One way to justify that the constant of proportionality k is $\frac{1}{3}$ is to ask pupils to make six identical square-based pyramids that will fit together to make a cube with their apexes meeting at the centre of the cube (Note 3). If you say that you want the net of each pyramid to fit onto one sheet of A4 paper then pupils initially have to decide on a sensible size for the base of the pyramid: a 6 cm by 6 cm square base is reasonable. Then pupils need to use Pythagoras' theorem to work out the height of the isosceles triangles needed for the sloping sides of the pyramid ($3\sqrt{2}$ cm), so that the final pyramid will have a height of 3 cm (Fig. 4). That way six of them, with their apexes meeting, will form a cube, the six bases of the pyramids forming the six faces of the cube. Pupils can work in groups of three to draw two of these nets each, accurately, and then cut them out and glue them to make six identical pyramids. These should fit together exactly to make a cube.

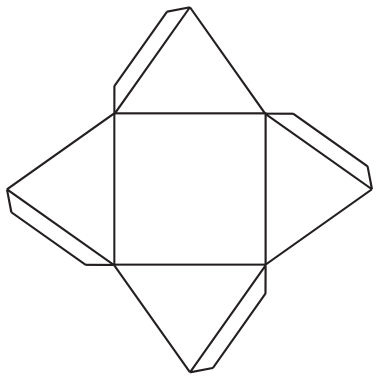

Fig. 4

(For a full-size accurate version of this net go to **www.foster77.co.uk/Net%20for%20a%20square-based %20pyramid.pdf.)**

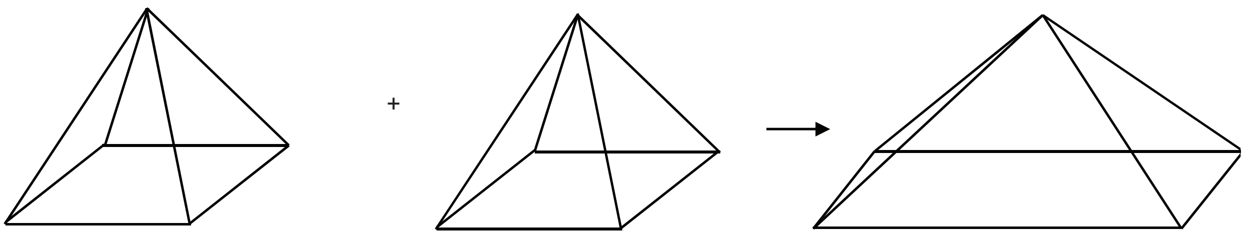

Fig. 2

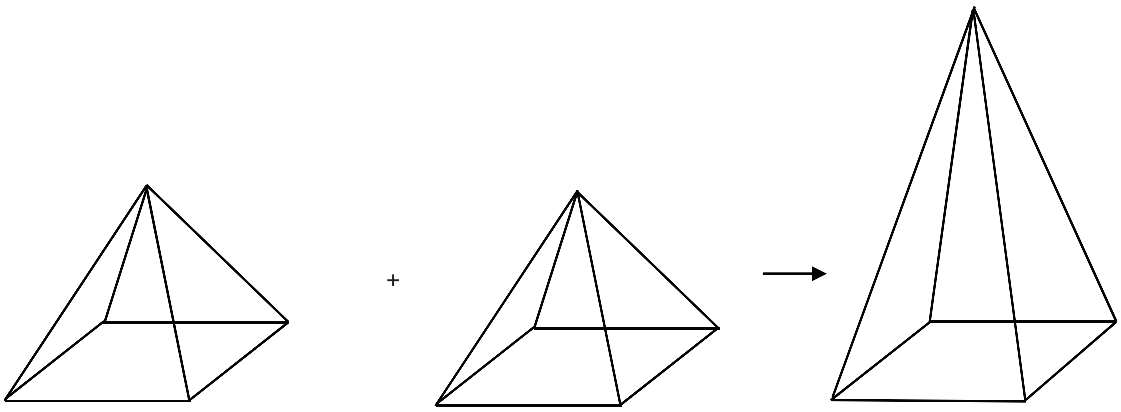

Fig. 3

This isn't rigorous unless we really justify why the pyramids form a cube without any tiny gaps, but it is highly suggestive. If we let the sides of the cube have unit length, so that the cube has unit volume, then each of the six pyramids must have a volume of $\frac{1}{6}$. We know that the base area of each pyramid is $1^2 = 1$, and the height is $\frac{1}{2}$, so $\frac{1}{6} = \frac{1}{2}k$, meaning that k must be $\frac{1}{3}$.

Now if we see a cone as a 'circular-based pyramid', having an n-gon base in the limit as n tends to infinity, then it is reasonable to see the formula $\frac{1}{3}\pi r^2 h$ as a special case of the formula for the volume of a pyramid, where the base area for a circular base is πr^2 (Note 4). So we have the volume of a pyramid and a cone.

By contrast with all this work, the *surface area* of a cone is much easier. We can split the curved surface area into lots of isosceles triangles with their bases on the circumference of the circular base and their apexes at the apex of the cone (Fig. 5). Each of these triangles has a tiny base b and a height equal to the slant height l (not the vertical height h) of the cone. Then the formula $\frac{1}{2}$ base × height gives the area of each little triangle as $\frac{1}{2}bl$, and, when we add up all of these, the b's will add up to the circumference of the base, which is $2\pi r$, so we get the curved surface area as $\frac{1}{2}(2\pi rl) = \pi rl$.

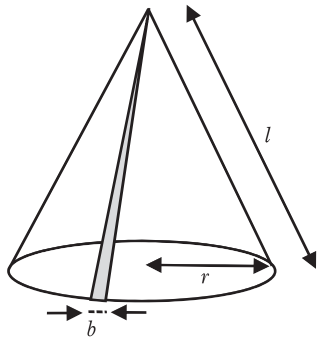

Fig. 5

Now all we have left to prove are the formulae for the volume and surface area of a sphere (Note 5). Perkins (2004) bases his demonstration on Archimedes' proof, which uses Cavalieri's principle which, putting it informally, states (in the 3D version) that if two shapes standing next to each other have equal cross-sections at every level, then they have the same volume. So we look at the cylinder, the double cone and the sphere shown in Figure 6. At every height h above or below the central horizontal plane, the shaded cross-sectional area of the double cone is πh^2, and the shaded cross-sectional area of the sphere is, using Pythagoras' theorem, $\pi(r^2 - h^2)$. The sum of these is πr^2, which is the shaded cross-sectional area of the cylinder. Since this is true in every horizontal plane, it follows that the *volume* of the cylinder must be equal to the volume of the double cone plus the volume of the sphere. So:

Volume of sphere

$$\begin{aligned} &= \text{volume of cylinder} - \text{volume of double cone} \\ &= \pi r^2 (2r) - \tfrac{1}{3}\pi r^2 (2r) \\ &= \tfrac{2}{3}\pi r^2 (2r) \\ &= \tfrac{4}{3}\pi r^3. \end{aligned}$$

Now we know the volume of a sphere, we can find its surface area quite easily. We just imagine the sphere to be made up of a large number of slender pyramids, of height r and small base area a, with all their apexes meeting at the centre (one pyramid is shown in Figure 7 – enlarged for clarity).

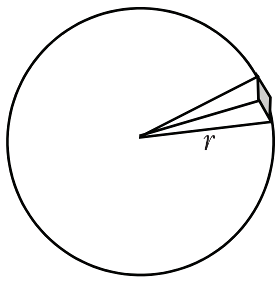

Fig. 7

Each little pyramid will have volume $\frac{1}{3}ar$, and when we add them all up we will obtain $\frac{1}{3}Ar$, where A is the total surface area of the sphere.

So $\frac{1}{3}Ar = \frac{4}{3}\pi r^3$, meaning that $A = 4\pi r^2$ (Note 6).

So we have some kind of justification for all of the volume and surface area formulae needed at GCSE. It's not the

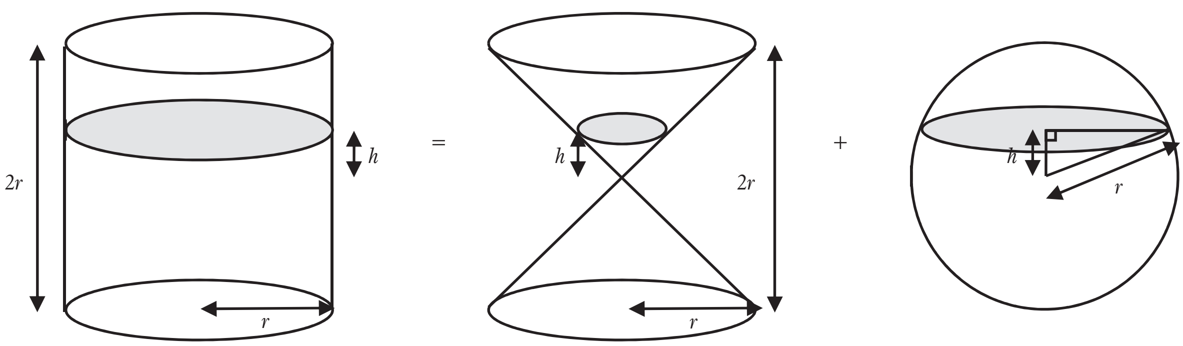

Fig. 6

last word on any of this, but at least it gives pupils a sense that these formulae don't just appear from nowhere, and that there is something here requiring explanation. It also prepares the way for more advanced treatments later.

Notes

1. In the new specification for the mathematics GCSE subject content (DfE, 2013), the formulae for volume and curved surface area of a cone, and volume and surface area of a sphere, come under the category of: "Formulae that candidates should be able to use, but need not memorize. These can be given in the exam, either in the relevant question, or in a list from which candidates select and apply as appropriate." (p. 16) By contrast, the formulae for the circle must be memorized. Formula posters are available at **www.cambridge.org/ukschools/files/3614/1503/1226/GCSE_Maths_Formulae_Infographic_PDF_Download_.pdf.**

2. In fact $3\pi r^2$ is the surface area of a solid *hemisphere*, which has a curved surface area of $2\pi r^2$ and a plane base area of πr^2.

3. Alternatively, you can construct three congruent square-based pyramids which fit together to make a cube, but this is harder to visualize: see **http://korthalsaltes.com/model.php?name_en=three+pyramids+that+form+a+cube.**

4. For a different approach to the original question, see Rowland (2012).

5. Strictly speaking (some would say), a sphere is a two-dimensional surface, and therefore has zero volume! The three-dimensional space contained *within* a sphere can be described as a *ball*. This is similar to the distinction between a circle (a one-dimensional curve) and a disc (a two-dimensional surface), so perhaps 'the area of a circle' should be 'the area of a disc'? For more on these kinds of ambiguities, see Foster (2011).

6. For a nice discussion of the fact that the surface area of a sphere is the derivative of the volume, just as the circumference of a circle is the derivative of the area, see Zazkis, Sinitsky & Leikin (2013).

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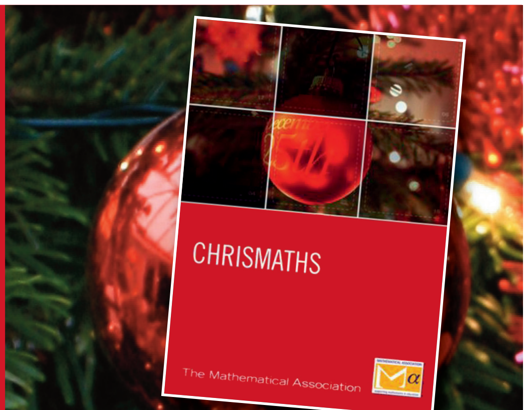

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